

## CHAPTER 13

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# FOURIER SERIES

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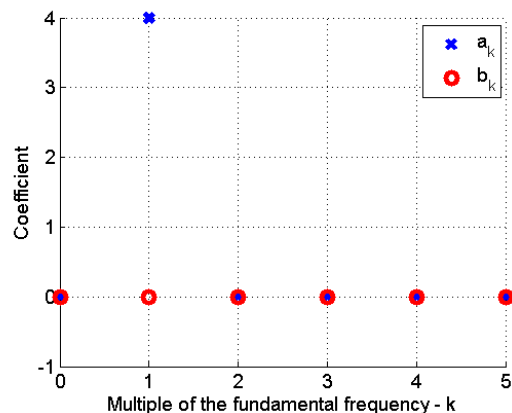
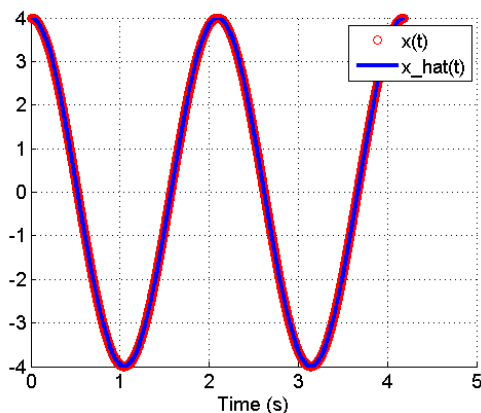
### 13.1 Trigonometric Fourier Series

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t) dt$$

### 13.2 Some Introductory Examples

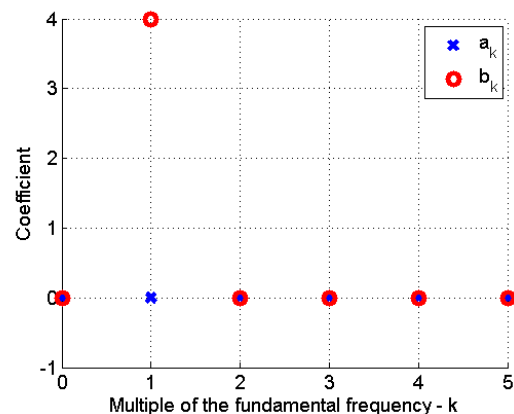
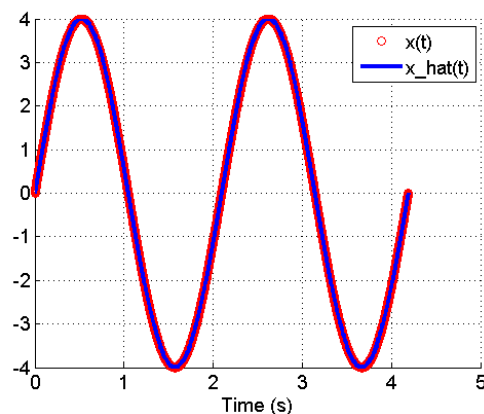


```
1 clear all
2 close all
3 clc
```

```

4
5 syms x w t T integrand
6
7 n=5; %# of coefficients to find
8 w0=3; %Frequency of the cosine
9 T0=(2*pi)/w0; % period that corresponds to w0
10 x=@(t) 4*cos(w0*t);
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
24 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro')
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)

```



```

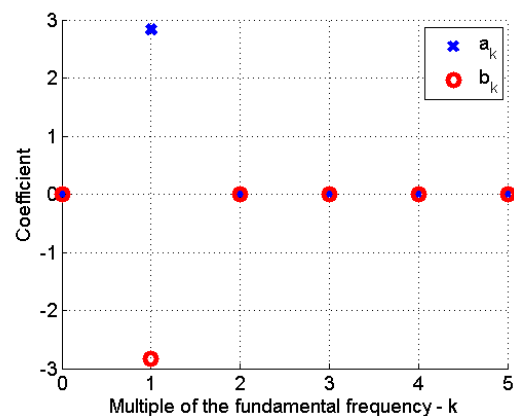
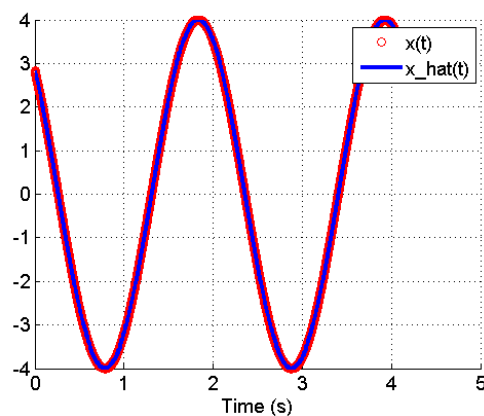
1 clear all
2 close all

```

```

3  clc
4
5  syms x w t T integrand
6
7  n=5; %# of coefficients to find
8  w0=3; %Frequency of the cosine
9  T0=(2*pi)/w0; % period that corresponds to w0
10 x=@(t) 4*sin(w0*t);
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
24 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro')
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x_hat(t)')
43 ylim([-4 4])
44 set(findall(gcf,'-property','FontSize'),'FontSize',14)

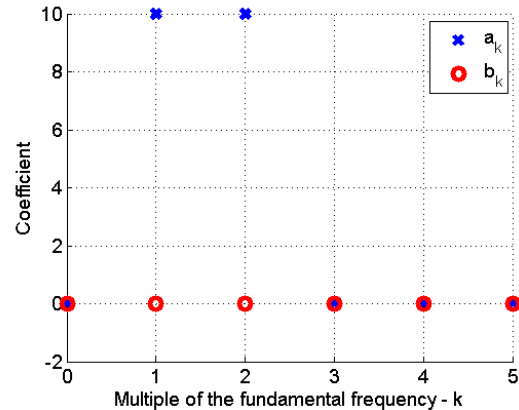
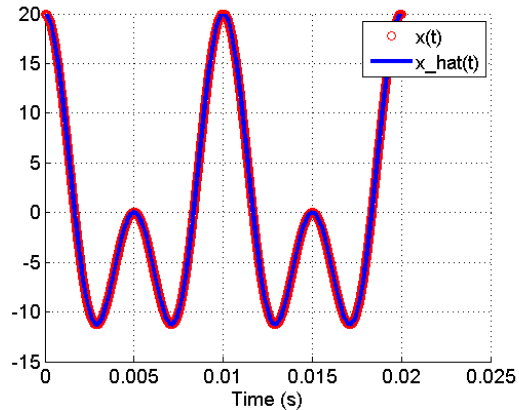
```



```

1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=5; %# of coefficients to find
8 w0=3; %Frequency of the cosine
9 T0=(2*pi)/w0; % period that corresponds to w0
10 x=@(t) 4*cos(w0*t+45*(pi/180));
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
24 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro')
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x\_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)

```

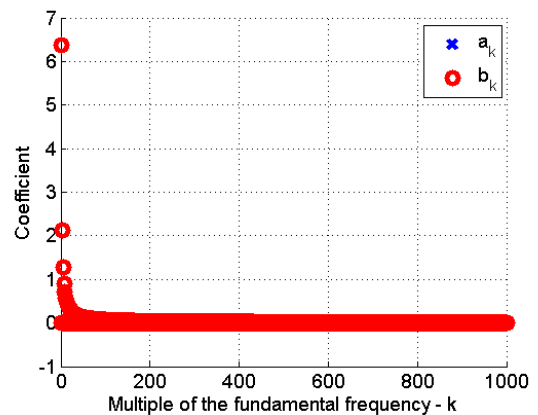
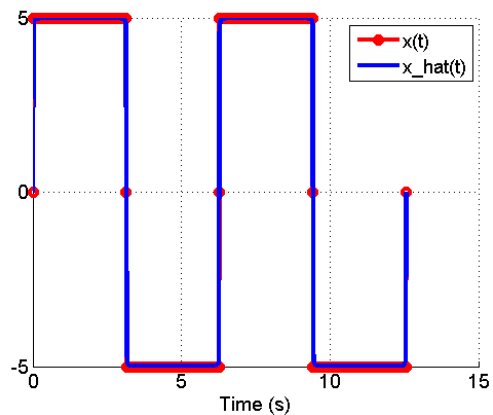
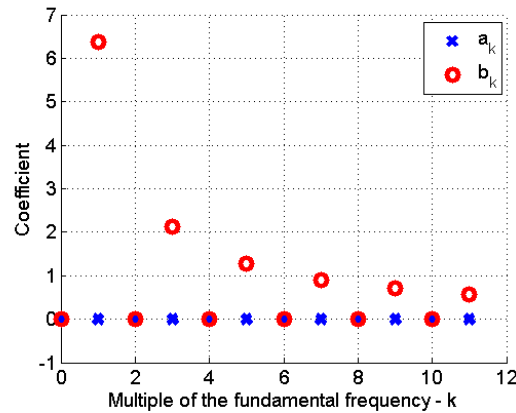
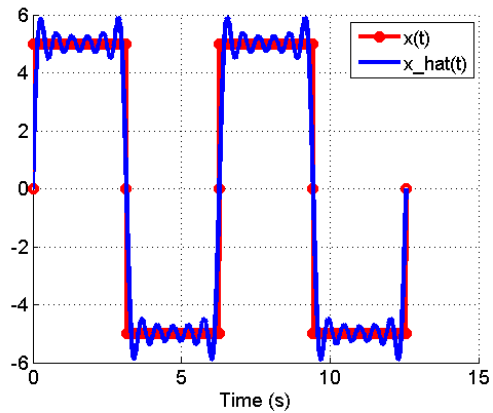
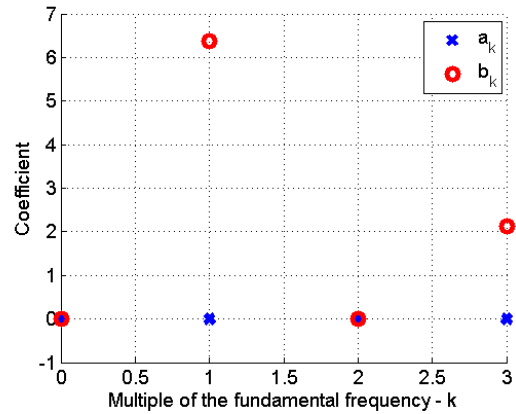
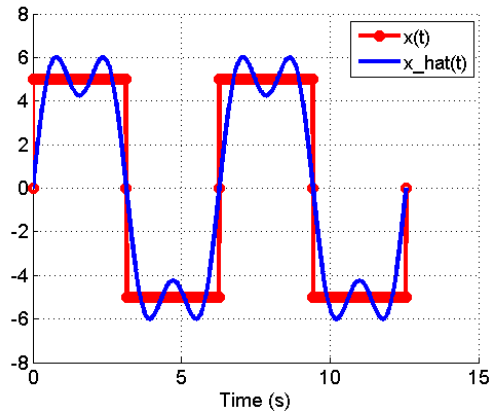
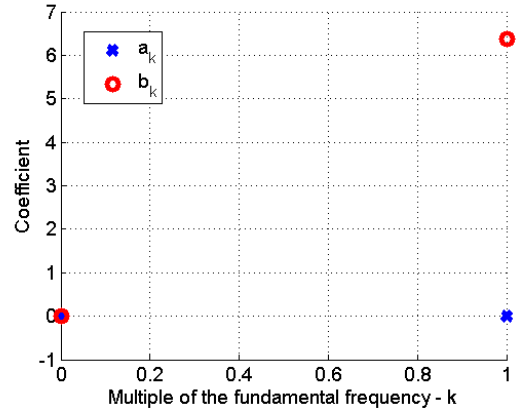
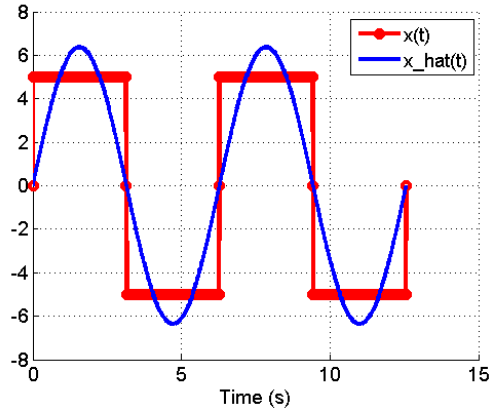


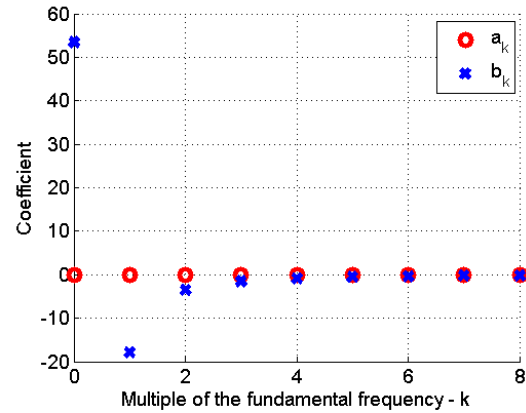
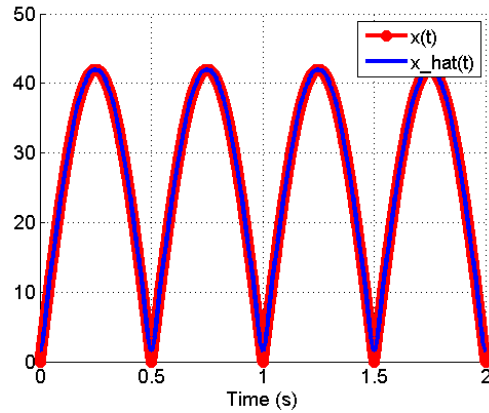
```

1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=5;
8 w0=628;
9 T0=(2*pi)/w0;
10 x=@(t) 10.*cos(w0.*t)+10.*cos(2.*w0.*t);
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
24 plot(0:length(b)-1,b,'ro','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro')
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)

```

```
1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=12;
8 w0=1;
9 T0=(2*pi)/w0;
10 x=@(t) 5*sign(sin(w0*(t)));
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 semilogy(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
24 semilogy(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro-','LineWidth',3)
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)
```





```

1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=8;
8 T0=1;
9 w0=(2*pi)/T0;
10 x=@(t) abs(42*sin(w0*t));
11 T0_rect=T0/2;
12 w0_rect=w0*2;
13 t0=0;t1=T0_rect;
14
15 for k=0:n;
16     integrand=@(t) (x(t)).*cos(k.*w0_rect.*t);
17     a(k+1)=(2/T0_rect)*integral(integrand,t0,t1);
18
19     integrand=@(t) (x(t)).*sin(k.*w0_rect.*t);
20     b(k+1)=(2/T0_rect)*integral(integrand,t0,t1);
21 end
22
23 figure(1)
24 hold on;grid on
25 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
26 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
27
28 xlabel('Multiple of the fundamental frequency - k')
29 ylabel('Coefficient')
30 legend('a_k','b_k')
31
32 tk=0:T0/1000:2*T0;
33 x_hat=(a(1)/2);
34 for k=1:n;
35     x_hat=x_hat+ a(k+1).*cos(k.*w0_rect.*tk)+ b(k+1).*sin(k.*w0_rect.*tk);
36 end
37 set(findall(gcf,'-property','FontSize'),'FontSize',14)
38
39 figure(2)
40 hold on;grid on
41 plot(tk,x(tk),'ro-','LineWidth',3)
42 plot(tk,x_hat,'LineWidth',3)
43 xlabel('Time (s)')

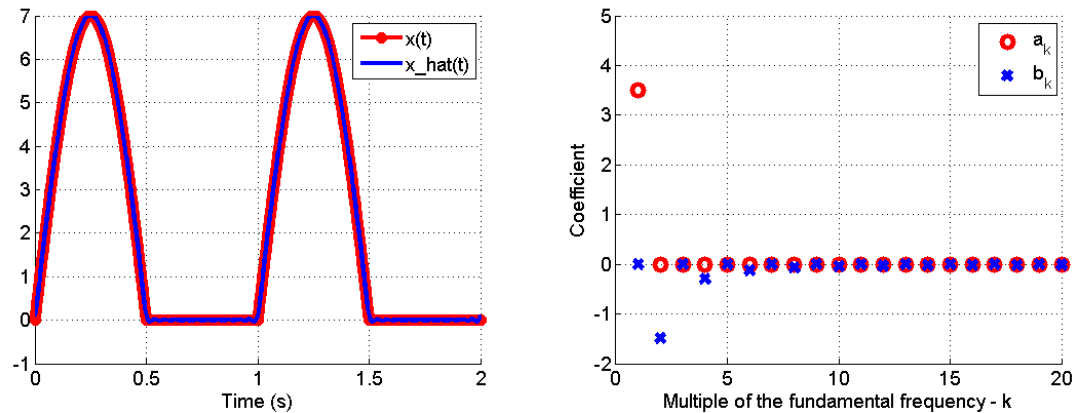
```



```

44 legend('x(t)', 'x_hat(t)')
45 set(findall(gcf, '-property', 'FontSize'), 'FontSize', 14)

```



```

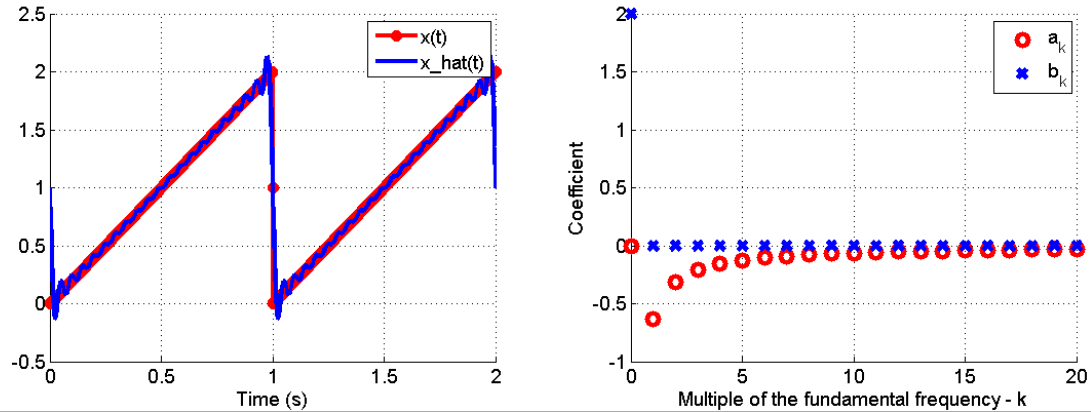
1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=20;
8 T0=1;
9 w0=(2*pi)/T0;
10 x=@(t) abs(7.*sin(w0*t)).*(heaviside((T0/2)-t)+heaviside(t-T0)-heaviside(t
    -(3*T0/2)));
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
24 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf, '-property', 'FontSize'), 'FontSize', 14)
36
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro-', 'LineWidth', 3)

```

```

40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)

```



```

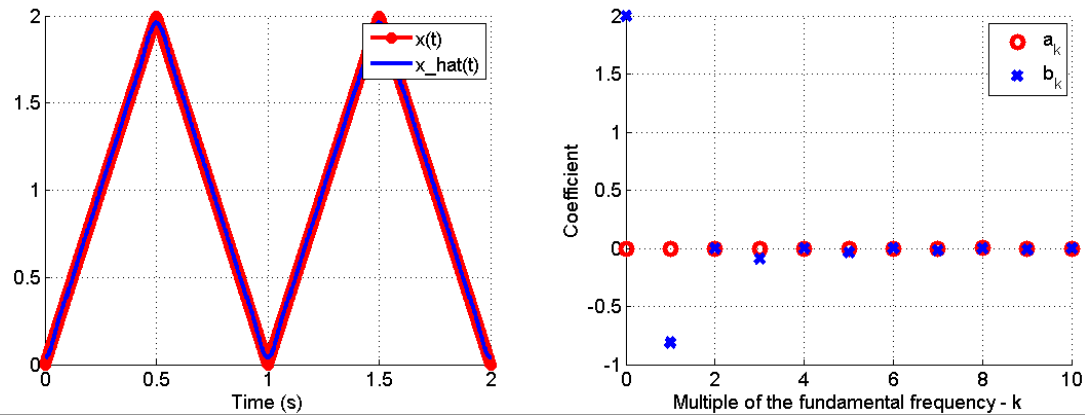
1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=20;
8 T0=1;
9 w0=(2*pi)/T0;
10 x=@(t) 2*t-2*heaviside(t-1);
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
24 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36
37 figure(2)
38 hold on;grid on

```

```

39 plot(tk,x(tk),'ro-','LineWidth',3)
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)

```



```

1 clear all
2 close all
3 clc
4
5 syms x w t T integrand
6
7 n=10;
8 T0=1;
9 w0=(2*pi)/T0;
10 x=@(t) 4*t.*heaviside(t)-8*(t-.5*T0).*heaviside(t-.5*T0)+8*(t-T0).*
    heaviside(t-T0)-8*(t-1.5*T0).*heaviside(t-1.5*T0);
11 t0=0;t1=T0;
12
13 for k=0:n;
14     integrand=@(t) (x(t)).*cos(k.*w0.*t);
15     a(k+1)=(2/T0)*integral(integrand,t0,t1);
16
17     integrand=@(t) (x(t)).*sin(k.*w0.*t);
18     b(k+1)=(2/T0)*integral(integrand,t0,t1);
19 end
20
21 figure(1)
22 hold on;grid on
23 plot(0:length(a)-1,b,'ro','LineWidth',4,'MarkerSize',10)
24 plot(0:length(a)-1,a,'x','LineWidth',4,'MarkerSize',10)
25
26 xlabel('Multiple of the fundamental frequency - k')
27 ylabel('Coefficient')
28 legend('a_k','b_k')
29
30 tk=0:T0/1000:2*T0;
31 x_hat=(a(1)/2);
32 for k=1:n;
33     x_hat=x_hat+ a(k+1).*cos(k.*w0.*tk)+ b(k+1).*sin(k.*w0.*tk);
34 end
35 set(findall(gcf,'-property','FontSize'),'FontSize',14)
36

```

```
37 figure(2)
38 hold on;grid on
39 plot(tk,x(tk),'ro-','LineWidth',3)
40 plot(tk,x_hat,'LineWidth',3)
41 xlabel('Time (s)')
42 legend('x(t)','x\_hat(t)')
43 set(findall(gcf,'-property','FontSize'),'FontSize',14)
```

### 13.3 Examples of Arbitrary Functions