CHAPTER 10

TRANSFER FUNCTIONS

In this chapter we will look at a another representation of a circuits frequency response. While the Bode plot is useful to visualize a circuits behavior over many decades of frequency, it is limited when more exact calculations are needed. A transfer function is closely related to a Bode plot in that it represents the complex gain of a circuit. If we know the transfer function of a circuit, we can relate an input voltage or current to the output voltage or current without knowledge of the circuit's schematic.

> **Transfer Function:**
>
> A function of complex frequency relating the input and output of a circuit. Evaluating the transfer function at a frequency yields the complex gain at that frequency.

10.1 Complex Frequency

The definition of a transfer function shown above refers to "complex frequency". The idea of complex frequency is an artifact of the differential equations underlying much of this material.

TO DO: Review Exponential form Replace j theta with jwt Graphic over time as t covers a period

So $j\omega$ shows up in the phasor representation of sinusoidal signals. $j\omega$ also appears in the impedance of inductors and capacitors.

$$Z_L = j\omega L \qquad\qquad Z_C = \frac{1}{j\omega C}$$

Given the two values are frequently paired together we will treat them as a single value, the complex frequency s, for the remainder of this book.

Let's reconsider the exponential form of representing phasors and the impedances of passive components using the substitution

$$s = j\omega$$

We will begin using the complex frequency representation of the following values as we develop the concept of transfer functions and perform analysis to find these functions.

	Type	Angular Frequency	Complex Frequency
Phasor	Voltage/ Current	$e^{j\omega t}$	e^{st}
Impedance	Resistor	R	R
	Inductor	$j\omega L$	sL
	Capacitor	$\frac{1}{j\omega C}$	$\frac{1}{sC}$

We're beginning to move away from analyzing a circuit at a single frequency and consider how a circuit behaves for all frequencies. Rather than individual phasors and impedances we will now have functions that represents the phasors and impedances with respect to complex frequency.

10.2 Using Transfer Functions

Let's begin by working with a transfer function and relating it to complex gain in a similar fashion to how we began working with Bode plots.

EXAMPLE 10.1

Given the transfer function

$$H(s) = \frac{1}{RCs+1} = \frac{1}{0.00265s+1}$$

Find the gain at 25 Hz, 60 Hz, 600 Hz

f=25 Hz

Let begin by converting f to ω

$$\omega = 2\pi f = 2\pi(25\ Hz) = 50\pi\ {}^{rad}/_{s}$$

Now that we have a value for ω we can evaluate the transfer function for the corresponding value of s. In this case, $s = j50\pi$

$$H(j50\pi) = \frac{1}{0.00265(j50\pi)+1} = 0.9232\angle{-22.6^\circ}\ {}^{V}/_{V}$$

Stop here and make sure you can evaluate the function as I have. Use whichever tool you will need to use in the future. A scientific calculator or MATLAB should do the trick.

The other frequencies are calculated in a similar fashion. Use the opportunity practice.

f=60 Hz

$$H(j120\pi) = \frac{1}{0.00265(j120\pi)+1} = 0.7075\angle{-44.97^\circ}\ {}^{V}/_{V}$$

f=600 Hz

$$H(j1200\pi) = \frac{1}{0.00265(j1200\pi)+1} = 0.0996\angle{-84.28^\circ}\ {}^{V}/_{V}$$

If you examine the magnitude of the gains as frequency increases, you may notice that the lower frequencies pass signals and the higher frequencies attenuate signals. We call this a low-pass filter because of this. More on this later.

EXAMPLE 10.2

A circuit has the transfer function

$$H(s) = \frac{RCs}{LCs^2+RCs+1}$$

and part values of

Part	Value
R	12.2k Ω
L	100 mH
C	0.01 μF

Find the gain for 1 kHz, 5 kHz, 20 kHz

Start by substituting the part values into the transfer function

$$H(s) = \frac{RCs}{LCs^2 + RCs + 1} = \frac{122e{-}6s}{1e{-}9s^2 + 122e{-}6s + 1}$$

Now we evaluate $H(s)$ for the three complex frequencies indicated

f=1 kHz

$$H(j2000\pi) = \frac{122e{-}6(j2000\pi)}{1e{-}9(j2000\pi)^2 + 122e{-}6(j2000\pi) + 1} = 0.623\angle 51.4^\circ \ {}^V\!/_V$$

f=5 kHz

$$H(j10000\pi) = \frac{122e{-}6(j10000\pi)}{1e{-}9(j10000\pi)^2 + 122e{-}6(j10000\pi) + 1} = 0.999\angle 0.195^\circ \ {}^V\!/_V$$

f=20 kHz

$$H(j40000\pi) = \frac{122e{-}6(j40000\pi)}{1e{-}9(j40000\pi)^2 + 122e{-}6(j40000\pi) + 1} = 0.720\angle{-44.0}^\circ \ {}^V\!/_V$$

Stop and use a calculator or MATLAB to confirm these values.

In the next example we'll use a transfer function in place of circuit analysis. We'll follow the same path we've used in analyzing circuits with phasors. We'll move to the phasor domain, use the transfer function to find the phasor domain output, and move back to the time domain.

EXAMPLE 10.3

Given a circuit with the transfer function

$$H(s) = \frac{122e{-}6s}{1e{-}9s^2 + 122e{-}6s + 1}$$

Find the output if the input is $v_{in} = 140cos(2000t + 10^\circ)\ mV$.

Moving $v_{IN}(t)$ to the phasor domain give us

$$V_{IN} = 140\angle 10^\circ \ mV$$

Next we evaluate $H(s)$ for $\omega = 2000\ {}^{rad}\!/_s$

$$H(j2000) = \frac{122e{-}6(j2000)}{1e{-}9(j2000)^2 + 122e{-}6(j2000) + 1} = 0.238\angle 76.2^\circ \ {}^V\!/_V$$

We need only to multiply the input by the gain to get the phasor domain solution

$$V_{OUT} = (140\angle 10^\circ \ mV)(0.238\angle 76.2^\circ \ {}^V\!/_V) = 33.3\angle 86.2^\circ \ mV$$

Finally, we move back to the time domain

$$v_{OUT}(t) = 33.3\cos(2000t + 86.2^\circ)\ mV$$

Just as with our first look at Bode plots, this is not an exhaustive treatment of transfer functions. It is only intended to link the material conceptually to the concept of gain as we studied it in previous chapters.

10.3 Finding Transfer Functions

Now that we can do some basic calculations with transfer functions we need to develop these functions for ourselves using circuit analysis. We use the same approach as we did when studying gain. Write an expression for the output and divide by the input. In the chapter on gain we performed the analysis to write the output for a single frequency. This allowed us to use the alternate approach of assuming an input value. Now that we are considering how the circuit behaves for all frequencies this alternate approach is no longer valid.

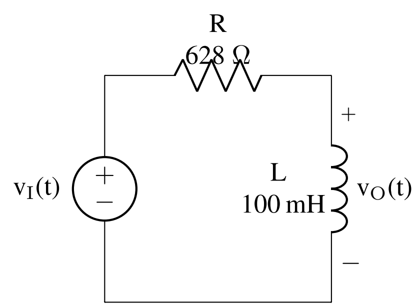
Previously we performed the analysis to write the output expression in the phasor-domain. Now we will perform the analysis in the closely related frequency-domain, sometimes called the s-domain. We saw the transformation to the s-domain earlier in this chapter but I will repeat the table of impedance here for easy reference.

Component	s-Domain Impedance
Resistor	R
Inductor	sL
Capacitor	$\frac{1}{sC}$

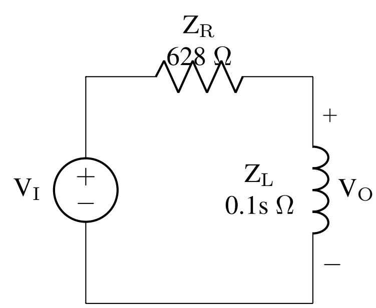
Reference this table as we work through the examples in the rest of this chapter.

EXAMPLE 10.4

Find the transfer function for this circuit



We begin by moving from the time-domain to the frequency-domain.



Notice that the circuit forms a voltage divider. We're starting with a simple circuit here as a first step. We can write an expression for the output

$$V_O = V_I \left[\frac{Z_L}{Z_L + Z_R} \right]$$

and divide by the input to find the gain

$$A_V = \frac{V_O}{V_I} = \left[\frac{Z_L}{Z_R + Z_L} \right]$$

Substituting the impedances into the gain equation gives us a function of complex frequency. We'll start using the functional notation we saw in the first examples in this chapter

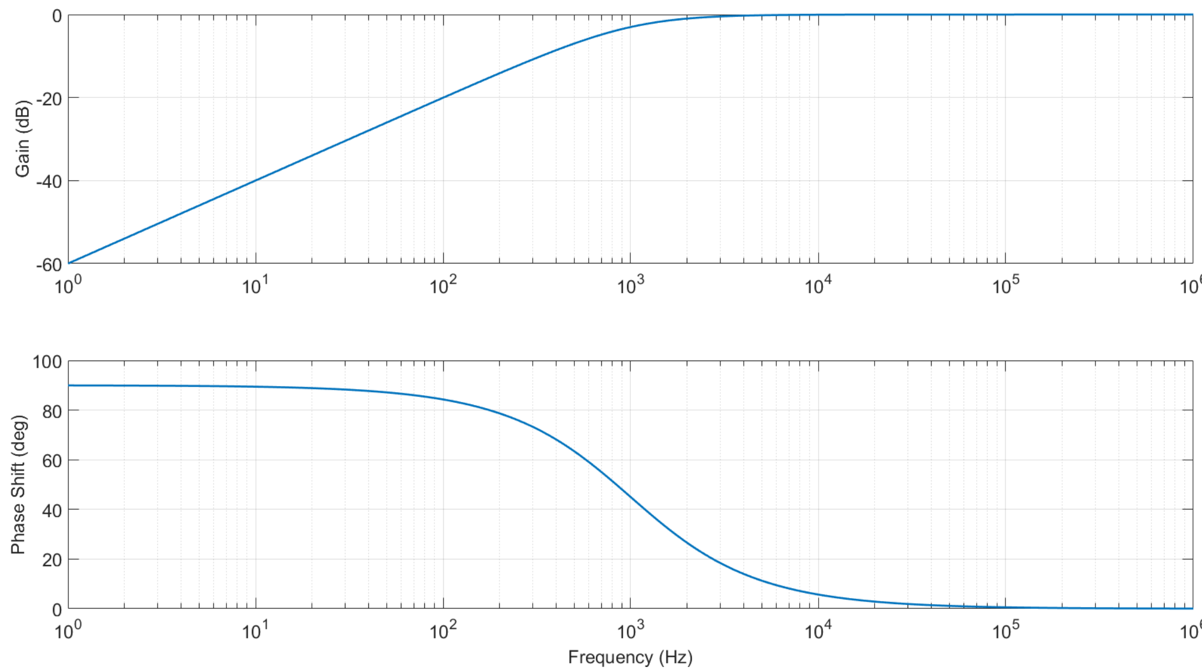
$$H(s) = A_V = \frac{0.1s}{0.1s + 628} = \frac{s}{s + 6280}$$

This transfer function is already in an acceptable format. It is a fraction of two polynomials. More complex circuits will require additional manipulation to get the transfer function into this format.

We can use $H(s)$ to find gains at individual frequencies or, using MATLAB's ability to perform many calculations quickly, find the magnitude and phase shift of the transfer function over many decades of frequency. Plotting the magnitude in decibels and the phase-shift gives us the Bode plot for this circuit. Take a look at the MATLAB code below to see how to generate the Bode plot.

```
%% Finding a transfer function with Voltage Divider
clear all
close all
clc
format short eng

syms R L s Vin Va Vb Vout

Zl=s*L;

H=Zl/(R+Zl);% Gain is output over input

H=subs(H,[R L],[628 100e-3]);
pretty(simplify(H)) %Print the transfer function in the Command Window
f=logspace(0,6,1000);
w=2*pi*f;
H=subs(H,s,j*w);

figure(1)
subplot(2,1,1)
semilogx(f,20*log10(abs(H)),'LineWidth',2)
grid on
fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)
ylabel('Gain (dB)')

subplot(2,1,2)
semilogx(f,angle(H)*(180/pi),'LineWidth',2)
grid on
xlabel('Frequency (Hz)')
ylabel('Phase Shift (deg)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)
```

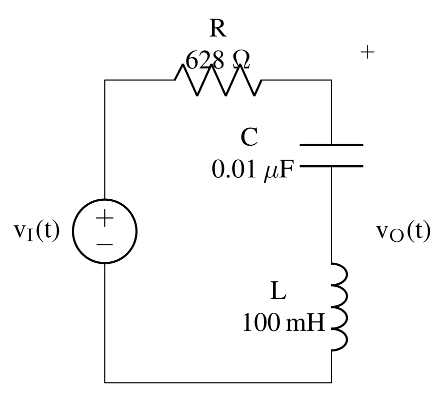
This code will generate the Bode plot for the circuit in this example. The Bode plot is shown here:



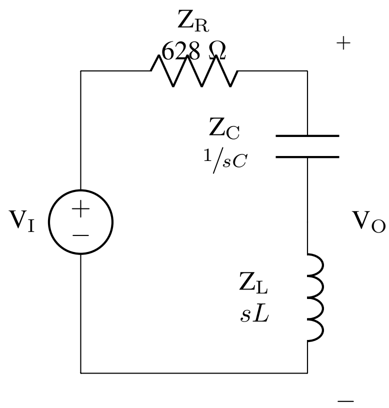
The next example requires a similar approach to circuit analysis to write an expression for the output. However, the resulting function will require more manipulation to get it into polynomial form.

EXAMPLE 10.5

Find the transfer function for this circuit:



Again, we start by moving the problem to the frequency domain.



This circuit also forms a voltage divider like the previous example. However, the output voltage is taken across the capacitor and inductor in series. The expression for the output voltage is

$$V_O = V_I \left[\frac{Z_L + Z_C}{Z_L + Z_C + Z_R} \right]$$

We divide the output by the input to find gain

$$A_V = \frac{V_O}{V_I} = \left[\frac{Z_L + Z_C}{Z_L + Z_C + Z_R} \right]$$

Substituting the impedances into the gain equation gives us a function of complex frequency. We'll start using the functional notation we saw in the first examples in this chapter

$$H(s) = A_V = \frac{sL + {}^1\!/_{sC}}{sL + {}^1\!/_{sC} + R}$$

Moving this to an acceptable form requires us to move sC out of the denominator in both polynomials. We can multiply both eth numerator and denominator by sC to accomplish this.

$$H(s) = A_V = \frac{sL + {}^1\!/_{sC}}{sL + {}^1\!/_{sC} + R} \left(\frac{sC}{sC} \right) = \frac{LCs^2 + 1}{LCs^2 + RCs + 1}$$

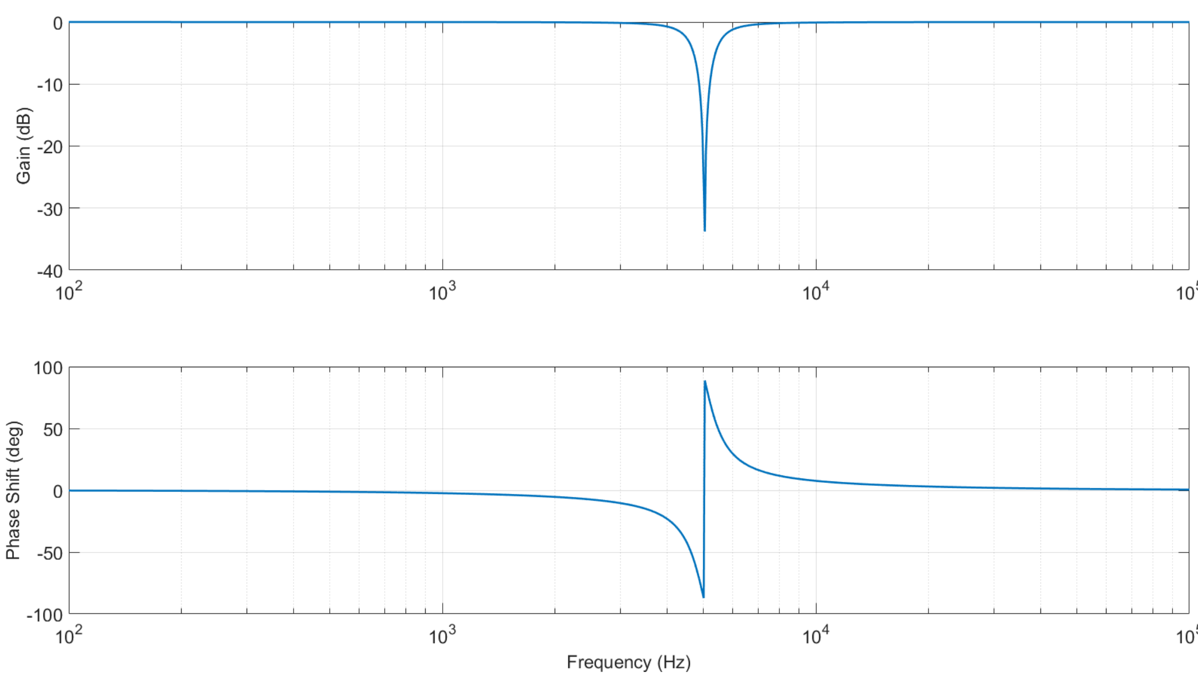
Finally, we substitute part values into the transfer function

$$H(s) = \frac{16s^2 + 80e6}{s^2 + 4s + 20e6}$$

The MATLAB code below performs this analysis and generates the Bode plot, also shown below.

```
%% Finding a transfer function with Voltage Divider
clear all
close all
clc
format short eng

syms R L C s Vin Va Vb Vout
Zl=s*L;
Zc=inv(s*C);

H=(Zl+Zc)/(R+Zl+Zc);% Gain is output over input

H=subs(H,[R L C],[628 100e-3 0.01e-6]);
pretty(simplify(H)) %Print the transfer function in the Command Window
f=logspace(2,5,1000);
```

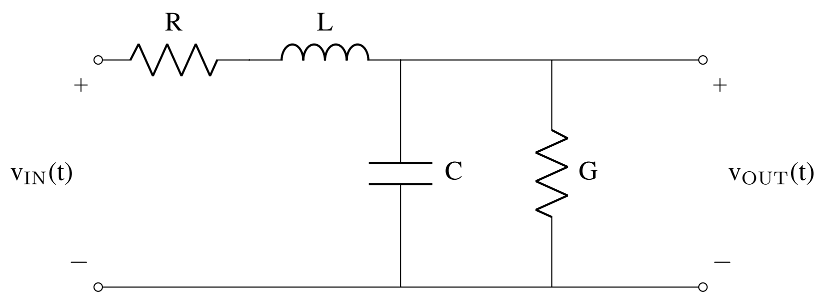
```
w=2*pi*f;
H=subs(H,s,j*w);

figure(1)
subplot(2,1,1)
semilogx(f,20*log10(abs(H)),'LineWidth',2)
grid on
ylabel('Gain (dB)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)

subplot(2,1,2)
semilogx(f,angle(H)*(180/pi),'LineWidth',2)
grid on
xlabel('Frequency (Hz)')
ylabel('Phase Shift (deg)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)
```



By now I'm hoping you're comfortable with the idea that a transfer function is just another representation of gain. I also hope that you are comfortable with the idea that the frequency-domain is closely related to the phasor-domain. Let's start working with more complex circuits. We'll begin with an example that we'll revisit once more in a later chapter.

EXAMPLE 10.6

The circuit we're going to analyze in this example is a model of a transmission line. With short lengths of cable we often neglect the resistance, inductance, and capacitance that form when two conductors are run in parallel. To analyze the effects of these values over a long distance cable we use the lumped component transmission line model shown below.



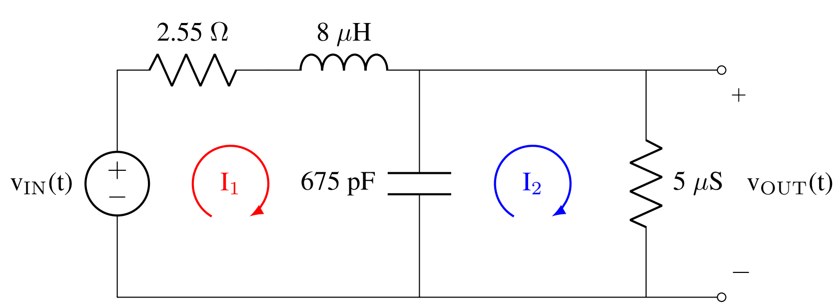
We're going to find the transfer function and plot the frequency response on a Bode plot. This is the beginning of understanding why there are limitations on baud rates we transmit digital data across such a transmission line.

You can find the properties of the cable that I mentioned above on a specification sheet or, if all else fails, by calling the manufacturer. There are also devices designed to measure these properties. I'll include typical values for a category 5 twisted pair for this analysis.

Component	Value
R	51 Ω/1,000 ft
L	160 μH/1,000 ft
C	13.5 nF/1,000 ft
G	100 μS/1,000 ft

Notice that the values are specified per unit of length. In this instance the manufacturer used an uncivilized unit (by that I mean a non-SI unit), the foot. Gigabit (1000BASE-T) ethernet is limited to 100 m (328 ft). You should also notice that G is a conductance specified in Siemens, a proper SI unit. We'll have to invert Let's consider a few lengths of cable to examine the effect of a longer transmission line. First, let's consider a short run of 50 ft. Then let's consider the maximum length specified for 1000BASE-T ethernet on a category 5 cable. I'll label the short length values on the schematic below and build a table for the two lengths we will consider. I'll also draw a source that will provide $v_{IN}(t)$.

Component	Value	50 ft	328 ft
R	51 Ω/1,000 ft	2.55 Ω	16.73 Ω
L	160 μH/1,000 ft	8 μH	52.48 μH
C	13.5 nF/1,000 ft	675 pF	4.43 nF
G	100 μS/1,000 ft	5 μS	32.8 μS



My primary purpose for including this problem is to introduce more complicated circuit analysis to our discussion regarding transfer functions. The study of transmission lines is just a nice practical application. To meet the primary goal of this example I will use mesh analysis to develop the expression for the output voltage. I've labeled the two mesh currents in the schematic above to help up write the two KVL equations needed to analyze this circuit.

$$\textbf{KVL } \mathbf{I_1} \qquad (R + Z_L + Z_C)I_1 + (-Z_C)I_2 = V_{IN}$$

$$\textbf{KVL } \mathbf{I_2} \qquad (-Z_C)I_1 + (Z_C + {}^1\!/_G)I_2 = 0$$

We can use MATLAB to solve the system of equations. Once we have a solution for I_1 and I_2 we can find the output voltage with

$$V_{OUT} = (^1/_G)\, I_2$$

and find the transfer function with

$$H(s) = \frac{V_{OUT}}{V_{IN}}$$

Next I'll use MATLAB to substitute the part values for both lengths of cable and generate the Bode plots for both cables.

```
%% Finding a transfer function with Mesh Analysis
clear all
close all
clc
format short eng

syms R G L C s Vin I1 I2
Zl=s*L;
Zc=inv(s*C);

eqn(1)=(R+Zl+Zc)*I1+(-Zc)*I2==Vin;
eqn(2)=(-Zc)*I1+(Zc+(1/G))*I2==0;
sol=solve(eqn,I1,I2);
Vout=sol.I2*(1/G);

H=Vout/Vin;% Gain is output over input

H=subs(H,[R G L C],[2.55 5e-6 8e-6 675e-12]); %50 ft values
%H=subs(H,[R G L C],[16.73 32.8e-6 52.48e-6 4.43e-9]); %328 ft values
pretty(vpa(simplify(H),3)) %Print the transfer function in the Command Window
f=logspace(0,10,1000);
w=2*pi*f;
H=subs(H,s,j*w);

figure(1)
subplot(2,1,1)
semilogx(f,20*log10(abs(H)),'LineWidth',2)
grid on
ylabel('Gain (dB)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)

subplot(2,1,2)
semilogx(f,angle(H)*(180/pi),'LineWidth',2)
grid on
xlabel('Frequency (Hz)')
ylabel('Phase Shift (deg)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)
```

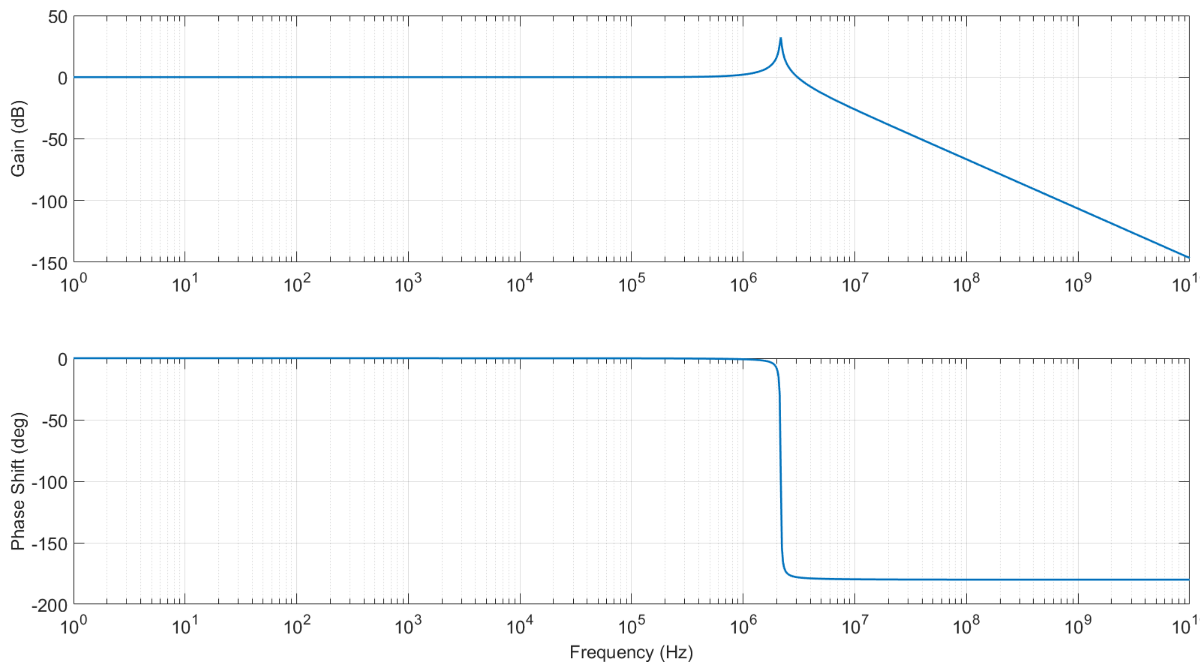
The transfer function for the 50 foot cable is

$$H(s) = \frac{V_{OUT}}{V_{IN}} = \frac{1}{5.4e{-}15s^2 + 1.76e{-}9s + 1}$$

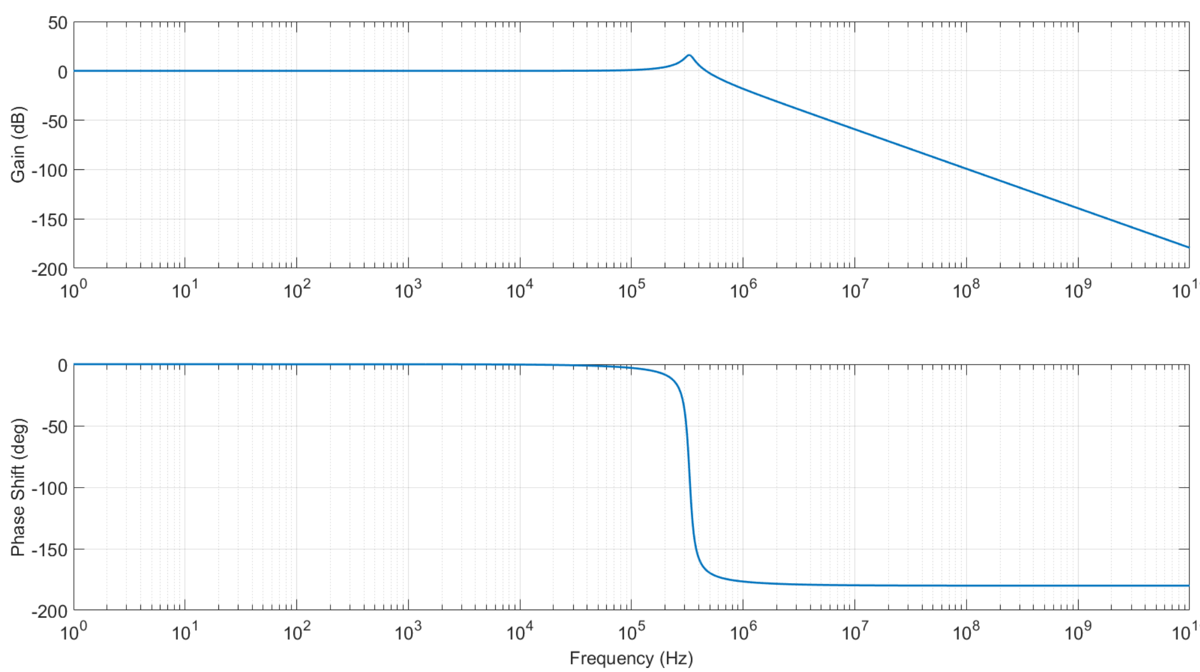
The transfer function for the 328 foot cable is

$$H(s) = \frac{V_{OUT}}{V_{IN}} = \frac{1}{232e{-}15s^2 + 75.8e{-}9s + 1}$$

Here is the Bode plot for the 50 foot cable. Notice two things. First, we cannot pretend that the cable does not exists. That is to say that we can no longer neglect the resistance, inductance, and capacitance of the cable. Perhaps for lower frequencies we can but not at the higher frequencies required by 1000BASE-T ethernet. Second, the cable begins to attenuate the signal around 3 MHz. We'll define the frequency at which a system begin attenuating a signal more formally in the next chapter. For now, a rough sense of this frequency is sufficient.



Here is the Bode plot for the 328 foot cable. Look for the frequency where attenuation begins. For this longer length of cable attenuation begins around 500 kHz preventing the higher frequency signal from reaching the receiving device on the network.

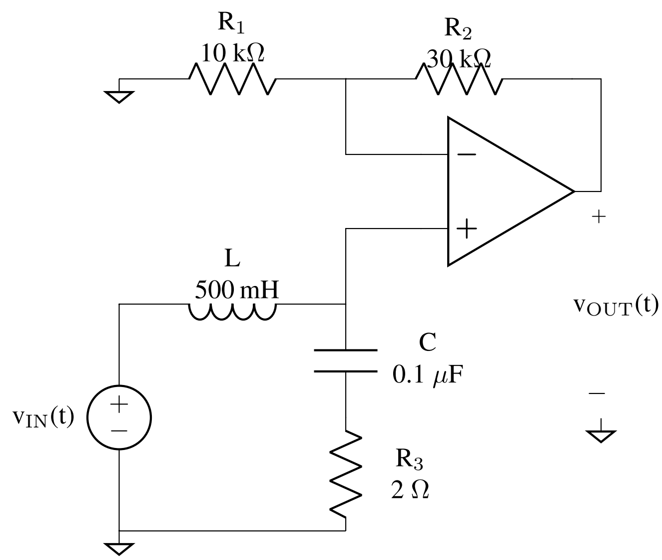


In a future chapter, we'll look at the signal attributes that contribute to this as well. For now, we're not equipped to begin that discussion. We'll also need a more formal definition of which frequencies pass through the cable and which do not. That formal definition is the subject of the next chapter.

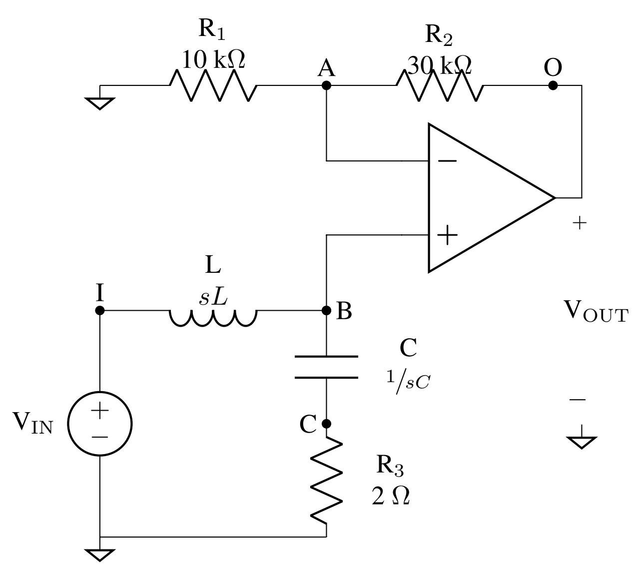
Let's look at one more example before we conclude this chapter. I hope this book enables you to analyze and circuit responding to any signal. In order to achieve this we need to include the use nodal analysis in finding transfer functions. Using nodal analysis lets us analyze many circuits that are otherwise difficult to analyze. Immediately, I'm thinking of circuits with operational amplifiers. In the future, you can use nodal analysis to analyze circuits with semiconductors.

EXAMPLE 10.7

Find the transfer function and generate the Bode plot for this circuit



This is a good opportunity to practice locating non-reference nodes. Take a look at the circuit above and try to count them. I'll label them in the next schematic. I'll also move the problem to the frequency-domain at the same time.



With the five non-reference (non-ground) nodes labeled we'll need five equations in the system of equations. Those five equations are shown here. The first is trivial and I've omitted it from the MATLAB code that follows. You can find the other four explicitly in the MATLAB code.

$$\begin{aligned}
\textbf{KVL I} &\qquad V_{IN} = V_{IN} \\
\textbf{KVL Op-Amp} &\qquad V_A = V_B \\
\textbf{KCL A} &\qquad -\frac{V_A}{R_1} - \frac{V_A - V_O}{R_2} = 0 \\
\textbf{KCL B} &\qquad \frac{V_{IN} - V_B}{Z_L} - \frac{V_B - V_C}{Z_C} = 0 \\
\textbf{KCL C} &\qquad \frac{V_B - V_C}{Z_C} - \frac{V_C}{R_3} = 0
\end{aligned}$$

The output equation is also trivial

$$V_{OUT} = V_O$$

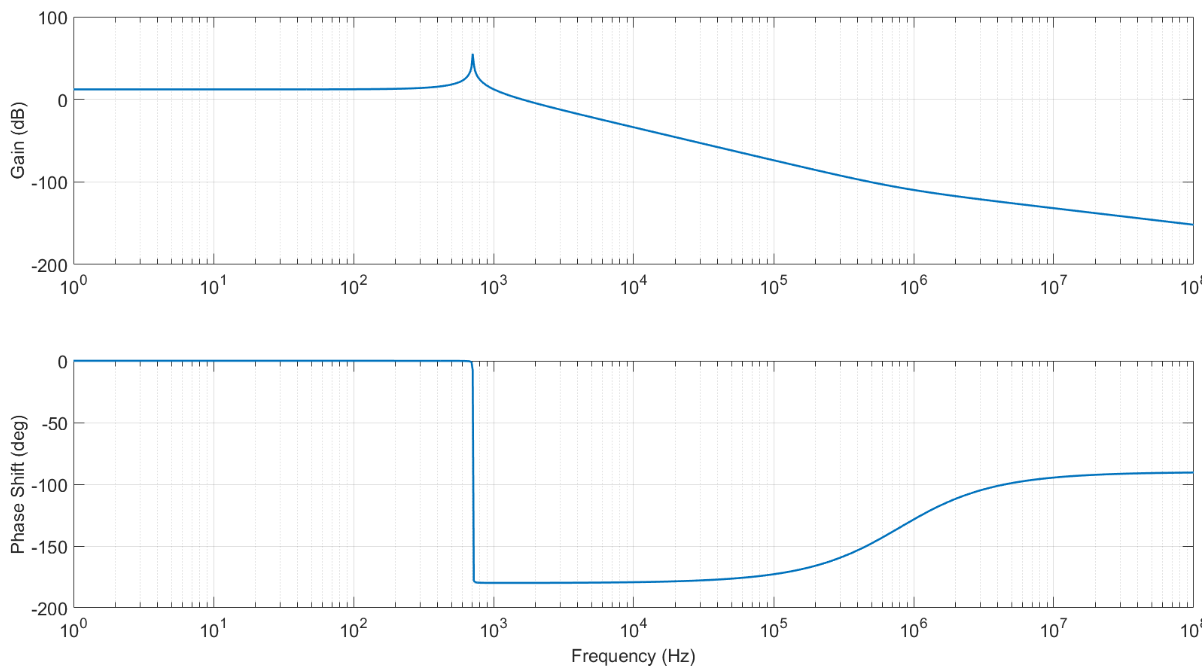
which leads to a transfer function

$$H(s) = \frac{V_{OUT}}{V_{IN}} = \frac{16s + 80000000}{s^2 + 4s + 20000000}$$

The MATLAB code and Bode plot are included here:

```
%% Finding a transfer function with Nodal Analysis
clear all
close all
clc
format short eng

syms R1 R2 R3 L C s Vin Va Vb Vc Vout

Zl=s*L;
Zc=inv(s*C);

eqn(1)=Va==Vb; %KVL Op Amp
eqn(2)=-((Va)/(R1))-((Va-Vout)/(R2))==0; % KCL Node A
eqn(3)=((Vin-Vb)/(Zl))-((Vb-Vc)/(Zc))==0; %KCL Node B
eqn(4)=((Vb-Vc)/(Zc))-((Vc)/(R3))==0; %KCL Node C
sol=solve(eqn,Va,Vb,Vc,Vout);

H=sol.Vout/Vin;% Gain is output over input

H=subs(H,[R1 R2 R3 L C],[10e3 30e3 2 0.5 0.1e-6]);
pretty(simplify(H)) %Print the transfer function in the Command Window
f=logspace(0,8,1000);
w=2*pi*f;
H=subs(H,s,j*w);

figure(1)
subplot(2,1,1)
semilogx(f,20*log10(abs(H)),'LineWidth',2)
grid on
ylabel('Gain (dB)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)

subplot(2,1,2)
semilogx(f,angle(H)*(180/pi),'LineWidth',2)
grid on
```

```
xlabel('Frequency (Hz)')
ylabel('Phase Shift (deg)')

fig=gcf;
set(findall(fig,'-property','FontSize'),'FontSize',18)
```



The last two examples demonstrate the utility of Mesh/Nodal analysis when paired with MATLAB. You have the tools to analyze any circuit to find its frequency response. If you can develop a system of equations (either mesh or nodal) for a circuit MATLAB will do the heavy-lifting of algebraically solving the system.

10.4 Cascaded Systems

TO DO

- cascaded systems
- loading effects
- isolation amplifier

CHAPTER 11

RESONANCE & FILTERS

11.1 Type of Filters

High/Low/Notch/Pass

11.2 First Order Filters

Analysis RC circuit 10 volt cos RC ckt 159 ohm/1microFarad

at 100 Hz calculate Zc, Zt,I point out magnitude of R vs Zc (looks more capacitive) highlight mag and angle of I

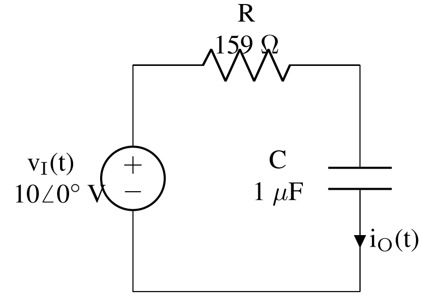
at 10kHz calculate Zc, Zt,I point out magnitude of R vs Zc (looks more resistive) highlight mag and angle of I

at w_c tau=RC w_c=1/tau f_c=w_c/(2*pi) calculate Zc, Zt,I draw phasor diagram point out Zc mag equals R (It's not resistive or capacitive, it's a balance between the two)

phase shift at cut-off

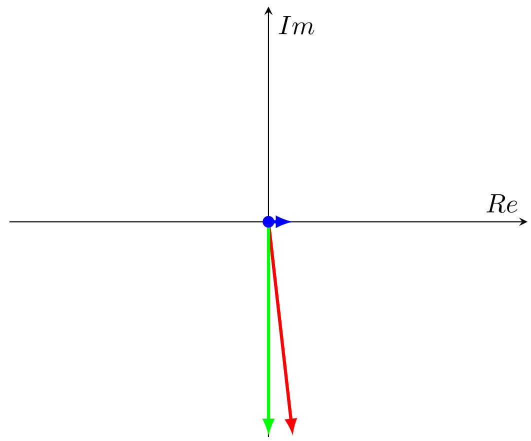
circuit to center freq and BW Q and BW

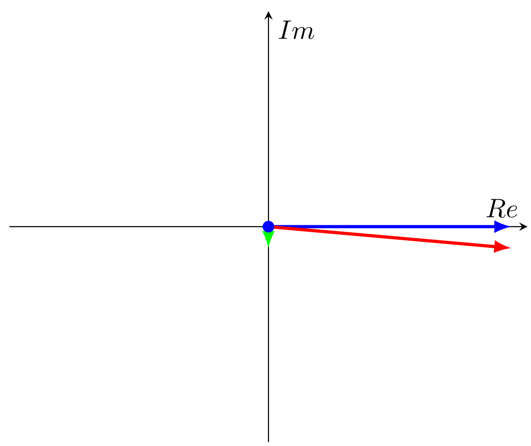
BW approximation/Large BW from crit freq, when to use which simple filter design passive and with op-amp

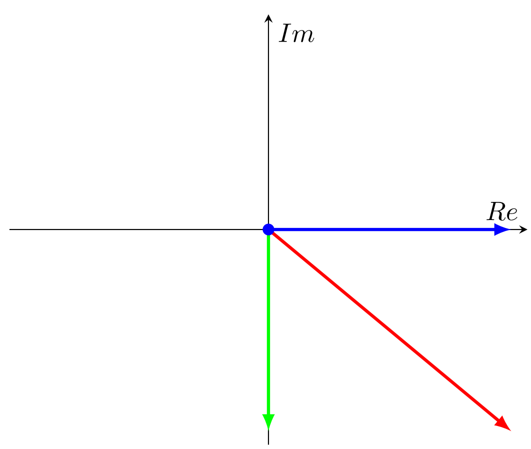


$$H(s) = \frac{Cs}{RCs + 1}$$

f	R	Zc	Z	I_o	A (V_O)	A ($_O$)
100 Hz	150 Ω	-j1.59 kΩ	150-j1.59 kΩ	$6.26\angle 84.6°$ mA	1	
10k Hz	150 Ω	-j15.92 Ω	150-j15.92 Ω	$66.3\angle 6.1°$ mA	0	
1.06k Hz	150 Ω	-j150 Ω	150-j150 Ω	$47.14\angle 45°$ mA	0.707	







Bode Plots for both

Pass-band: The range of frequencies between two cut-off frequencies that allow the signal to pass through an electrical system

Upper cut-off frequency: The frequency **above** which the output power of a circuit has fallen below the power of the pass-band

Lower cut-off frequency: The frequency **below** which the output power of a circuit has fallen below the power of the pass-band

11.3 Resonance in RLC Circuits

Bandwidth: The distance between cut-off frequencies. The width of the pass-band.

11.4 Estimating Bode Plots

$$H(s) = \frac{1}{RCs+1}$$

R=1 kΩ C=159 nF

```
clear all
close all
clc
format short eng

syms s
N=[1];  %coefficients of the numerator
D=[1e3*159e-9 1]; %coefficients of the denominator
zeros=abs(roots(N)); %rad/s
poles=abs(roots(D)); %rad/s

zeros/(2*pi) %convert to Hz
poles/(2*pi) %convert to Hz
```

$$H(s) = \frac{\frac{L}{R}s}{\frac{L}{R}s+1}$$

R=628 Ω L=100 mH

```
clear all
close all
clc
format short eng

syms s
N=[100e-3/628 0];  %coefficients of the numerator
D=[100e-3/628 1]; %coefficients of the denominator
zeros=abs(roots(N)); %rad/s
poles=abs(roots(D)); %rad/s

zeros/(2*pi) %convert to Hz
poles/(2*pi) %convert to Hz
```

$$H(s) = \frac{10000}{0.012s^2+1.42s+10000}$$

```
clear all
close all
clc
format short eng

syms s
N=[10000];  %coefficients of the numerator
D=[.012 1.42 10000]; %coefficients of the denominator
zeros=abs(roots(N)); %rad/s
poles=abs(roots(D)); %rad/s
```

```
zeros/(2*pi) %convert to Hz
poles/(2*pi) %convert to Hz
```

11.4.1 Magnitude

11.4.2 Phase Shift

11.5 Designing a Filter