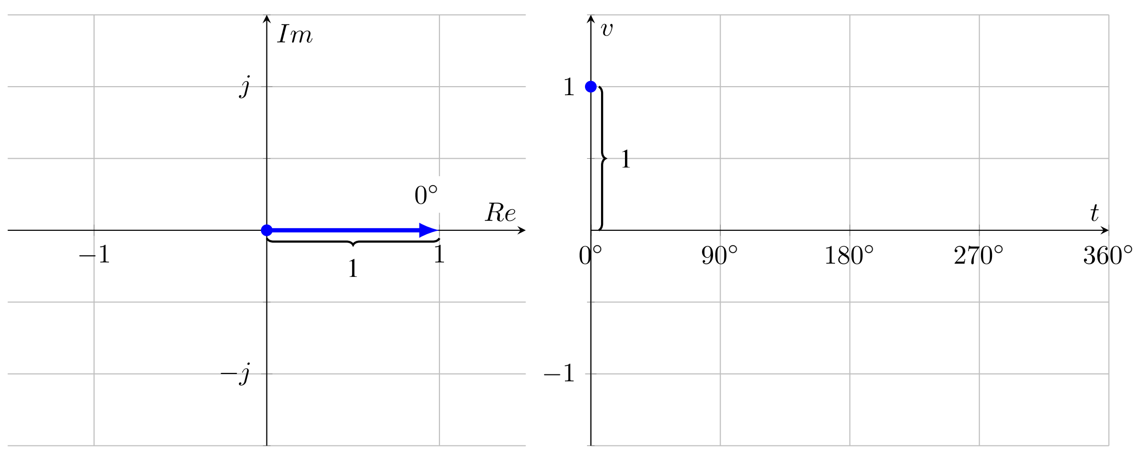
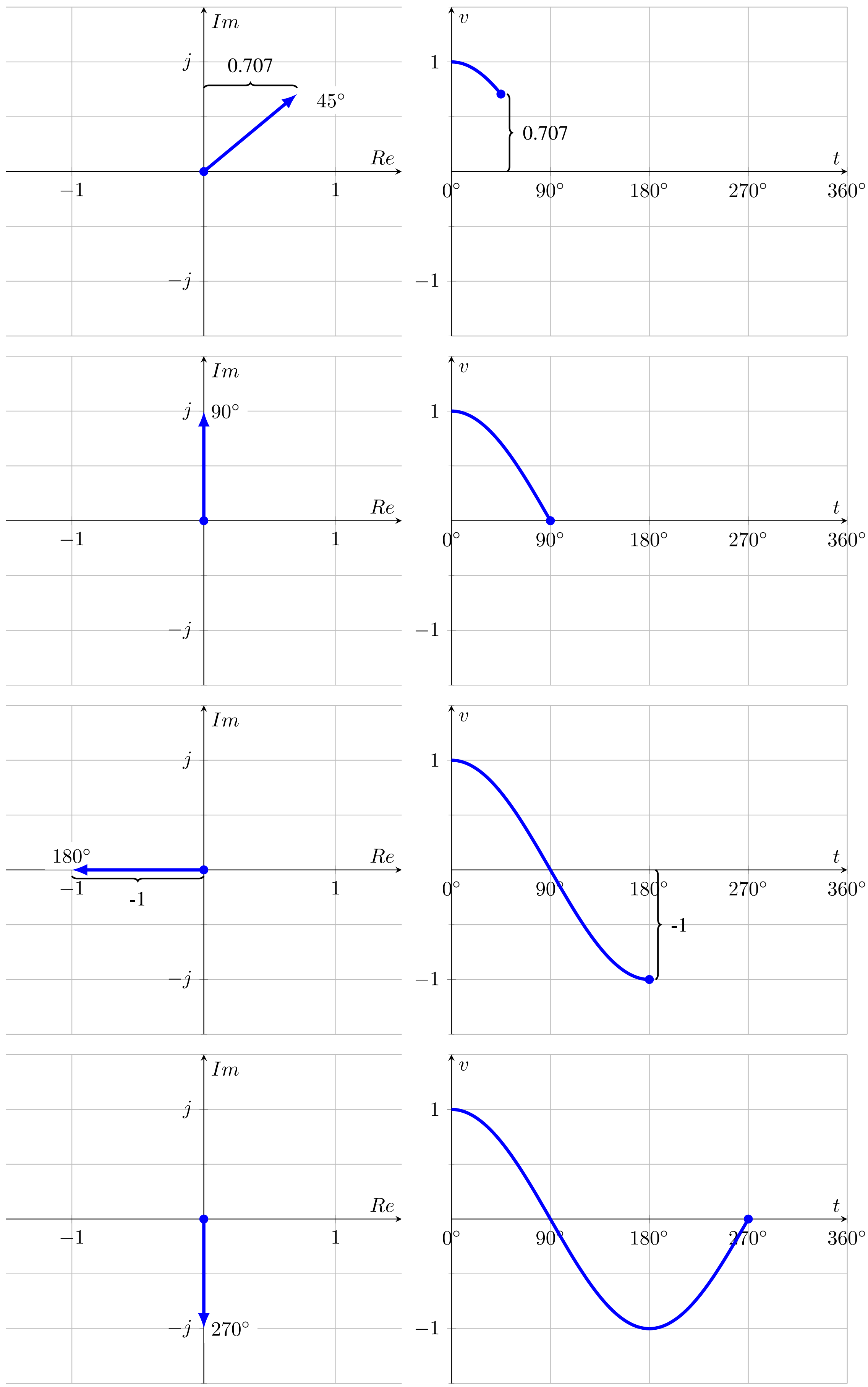
# CHAPTER 3

# REVIEW OF PHASORS

## 3.1 Alternating Current: Phasors

### 3.1.1 Sinusoidal Functions as Rotating Vectors





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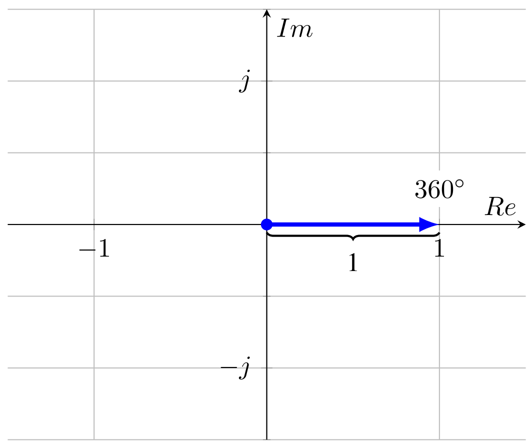
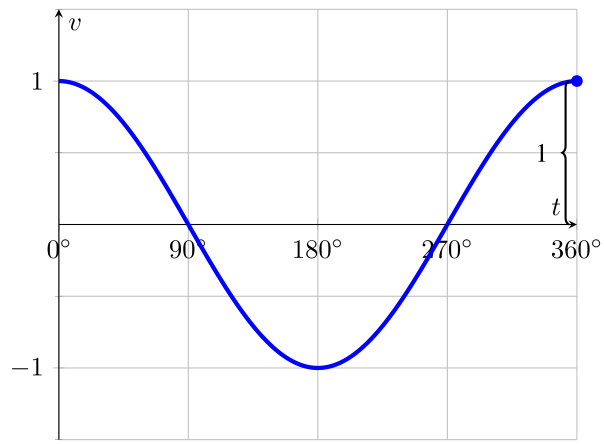


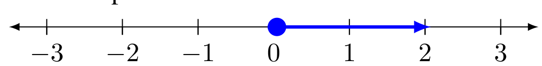




### 3.1.2 Review of Complex Numbers

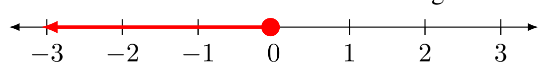
The first thing you learned about numbers was how to count: 1,2,3,4,... Then you learned what 0 was and then that there are numbers less than zero: -1,-2,-3,-4,... All of these numbers allow you to locate points on a numbers line. The numbers indicate two things:

1. How many "steps" or units away from the origin, indicated by the magnitude of the number
2. The direction away from the origin, indicated by the sign of the number

For instance, the number +2 locates a point on a 1D number line that is 2 units to the right of the origin




and the number -3 locates a point that is three units to the left of the origin

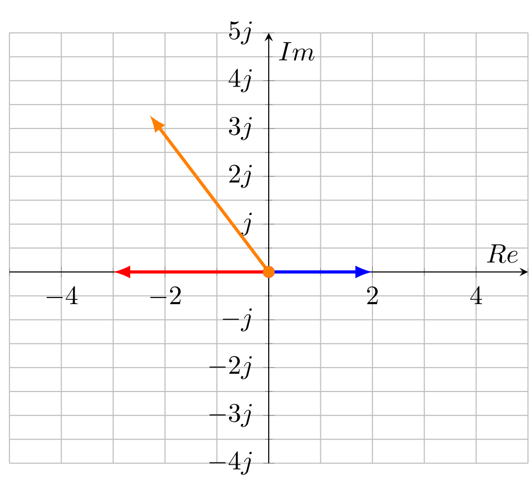



These numbers are often referred to as scalars.

A complex, or "imaginary", number accomplishes the same thing as a scalar number. It locates a point in space by giving an indication of the distance from the origin, and the direction away from the origin. However, there is a difference. A scalar locates a point on 1D line and a complex number locates a point on a 2D plane.

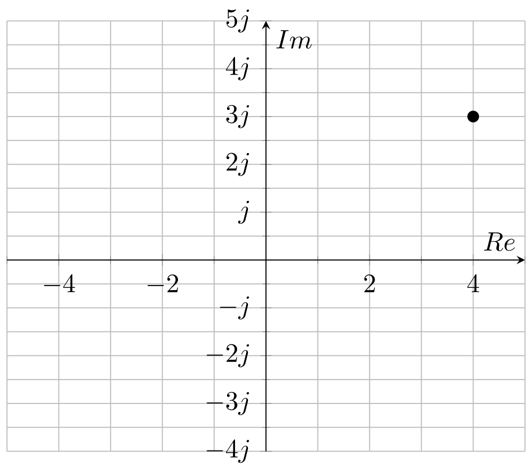
For example, the number +2 can be expressed as a complex number by indicating the distance from the origin and direction in a type of complex number called a polar number. The scalar +2 is equal to $(2\angle 0°)$ in polar form. The value before the angle symbol indicates the distance from the origin and the angle indicates the direction away from the positive real axis. A positive angle is measured in the counter-clockwise direction. $(2\angle 0°)$ is shown on the complex plane below in blue. -3 can be represented in polar form as $(3\angle 180°)$ and is shown in red.

The real advantage of complex numbers is the ability to locate numbers that are not on the real number line. For instance, to locate a point 4 units from the origin in the 125° direction we can draw $(4\angle 125°)$ in orange.



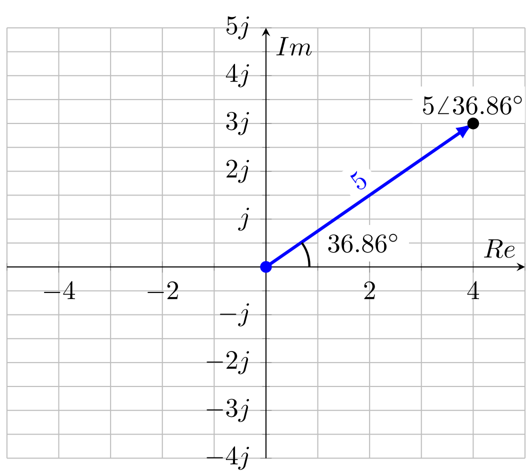
In all instances we are locating a point with a distance from the origin and a direction. This is consistent whether we are locating the point on a number line or the 2D complex plane.

***3.1.2.1 Other Complex Number Forms*** The form we have been using so far in this section is known as polar form. There are two other commonly used forms for complex numbers . In all cases the complex number represents a point on the complex plane. So let's pick a point on a plane:

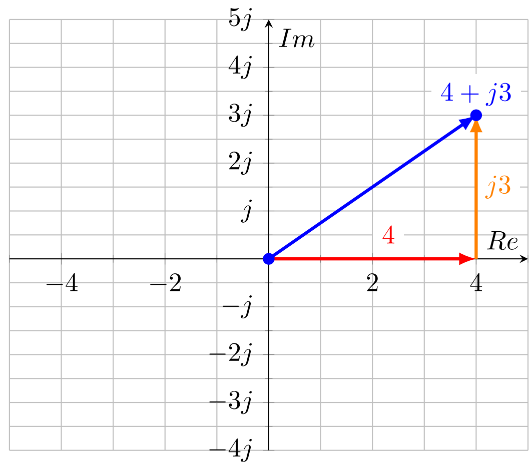



We can represent the location of the black dot shown on the plane above in three forms.

*Polar Form* This is the form introduced earlier in this section. It specifies a direction and distance from the origin to the point.




*Rectangular Form* We can use the rectangular form to represent the same point. This form specifies a distance from the origin along the real (horizontal) axis and a distance from the real axis parallel to the imaginary (vertical) axis.




The same type of information is included in this form: direction and distance. Direction is taken from the real and imaginary axes and distance is specified as scalar values. Ultimately we use this new form to locate the same point on the complex plane. Therefore we can correctly state that

$$5\angle 36.86° = 4 + j3$$

The complex numbers look different but they locate the same point on the plane.

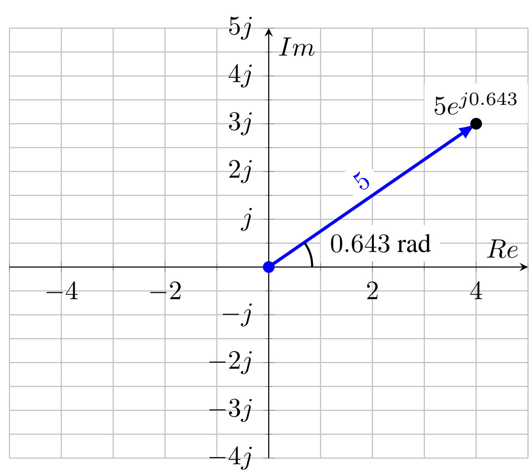
*Exponential Form* The last form we will discuss if the exponential form. This form shares the same information as polar but in a different format. The format change can be explained by a Swiss mathematician named Euler. I'll restate his identity here. I encourage you to read more about his work if you are interested but will not require you to understand the details of the identity.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

A coefficient is commonly included when using this form and can be distributed to the other terms

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta$$

Euler specified the angle of the complex value using radians rather than degrees. In the example we've been using $36.86°$ is approximately $0.643$ radians. Therefore we can locate the same point on the plane shown below




Notice that the information contained with in the exponential form $5e^{j0.643}$ contains a distance from the origin and an angle just as in the polar form. We can correctly state that

$$5\angle 36.86° = 5e^{j0.643}$$

While the identity originally specified the angle in radians modern calculators sometimes can specify the angle in degrees. So we may also see the exponential form written as

$$5e^{j36.86°}$$

*Conversion Between Complex Forms* Since all of the complex forms introduced above can specify the same point it is possible to convert any complex form to any of the others. If we reexamine the complex planes in this section it should become clear that the vectors of the different forms create a right triangle. Therefore, conversion between the types is an application of trigonometry and Pythagoras' theorem.

*Polar/Exponential to Rectangular* If a polar value is given as

$$A\angle\theta$$

conversion to exponential form is as simple as writing

$$Ae^{j\theta}$$

respecting the units of the angle $\theta$. Either of those two forms can be converted to the rectangular form

$$a + jb$$

using trigonometry to state

$$a = A\cos\theta$$

and

$$b = A\sin\theta$$

again respecting the units of $\theta$. (Check your calculator mode!)

*Polar/Exponential to Rectangular* if a rectangular value is given as

$$a + jb$$

it can be converted to either the polar or exponential forms

$$A\angle\theta = Ae^{j\theta}$$

using trigonometry and the Pythagorean theorem to state

$$A = \sqrt{a^2 + b^2}$$

and

$$\theta = \tan^{-1}\frac{b}{a}$$

once more, respecting the units of $\theta$. (Check your calculator mode again... just to be sure!) You must also consider that the angle $\theta$ is measured from the real axis in the counter-clockwise direction. If the complex number is in the second or third quadrant you must find the supplementary angle of theta before writing the converted complex value.

***3.1.2.2 Extending Operations into 2D Plane*** Now that we can locate points on a 2D plane instead of a simple 1D number line we should reconsider the operations we perform on these new numbers. What we will find is that the operations are the same regardless of the type of number we are using. Stated differently, the scalar operations reviewed below are equivalent to complex operations on a 2D plane.

Different operations are easier in different complex forms. When performing operations by hand I suggest converting the number to the following forms:

| **Rectangular** | **Polar or Exponential** |
|---|---|
| Addition | Multiplication |
| Subtraction | Division |
| Negation | Negation |
| | Inversion |

*Addition* With operations we can begin to visualize and represent more realistic problems. Since we're relearning how to add let's revisit some problem you may have heard when you were learning addition the first time such as

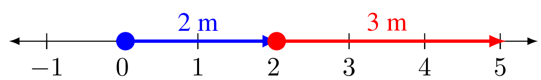
> *Colin walks 2 meters in one direction. Now he walks 3 meters in the same direction. How far is Colin from where he started?*

or

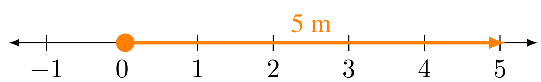
> *Lauren has 2 apples and Anna gave her 3 apples. How many apples does Lauren have?*

These problems are simplistic but allows us to illustrate the familiar operations we learned early in life. The problems also hint at the fact that these number represent something; physical distance in the first example and number of apples in the second. This seems natural to us when using scalar numbers. The same is true with complex numbers: the numbers still mean something and should have an appropriate unit assigned. Complex numbers can represent physical distance in the same way as scalars just with another dimension. Complex numbers can also represent voltage, current, force, torque, apples, etc.

Let's consider the first problem by showing the implied addition operation on a number line. The first move to the right can be represented by +2 and the second can be represent by +3. To add them together (2+3=5) on the number line we draw both vectors placing them "tip-to-tail".




The result of the operation is the location that the combination points to, in this case +5.

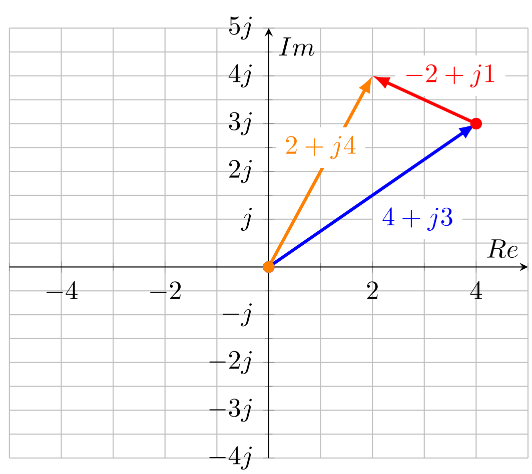



If we must express these quantities as complex numbers the imaginary components will be zero indicating that the number are on the real axis. The operation can be described as

$$\begin{array}{rr} & 2+j0 \\ + & 3+j0 \\ \hline & 5+j0 \end{array}$$

The result, $5 + j0$, can be rewritten as simply 5 since $0 \times j = 0$

We can take the same approach in the complex plane. Given two rectangular complex numbers, say $4 + j3$ and $-2 + j1$, we can add them together by placing them tip-to-tail on the complex plane.




The result is shown in orange on the plane and is equal to a vector with value 2+j4. We can arrive at this result mathematically rather than relying on the picture. In this case the real components add separately from the imaginary components.

$$\begin{array}{rr} & 4+j3 \\ + & -2+j1 \\ \hline & 2+j4 \end{array}$$

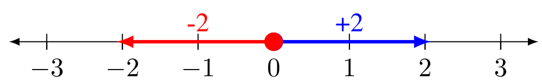
This operation is straight-forward if the complex values are in rectangular form. That is less true if they are in polar or exponential form. The plot above does not change but the calculation is less clear

$$\begin{array}{rr} & 5\angle 36.86^\circ \\ + & 2.236\angle 153.43^\circ \\ \hline & 4.472\angle 63.43^\circ \end{array}$$

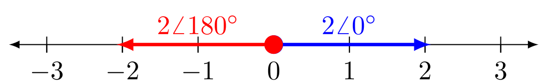
This is why I prefer rectangular form for addition.

*Negation* A scalar value has the same two pieces of information as a polar number. The distance from the origin is clearly the magnitude (absolute value) of the number. The direction may not be immediately obvious. The sign of the scalar number tells us whether it is located to the left or right of the origin.

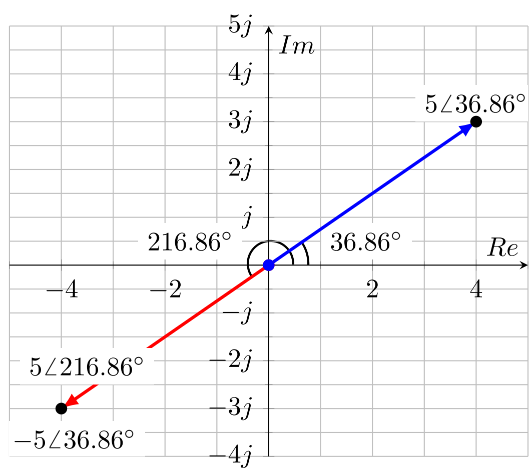
When we negate a number (multiply by -1) we simply switch the side of the origin the numberis located on. The distance from the origin remains the same.




We can also consider these numbers in polar form. We are constrained to the real numbers for now. The negative sign can be replaced with the angle as we would if the numbers were not constrained to the real axis.




So $-2\angle 0^\circ$ is equivalent to $2\angle 180^\circ$ and $-2\angle 180^\circ$ is equivalent to $2\angle 0^\circ$. This rotation by $180^\circ$ extends to the complex plane. Negating $5\angle 36.86^\circ$ is shown graphically here resulting in $5\angle 216.86^\circ$




Mathematically we can find the negative of a complex number in either form. In polar form, we rotate by $180^\circ$ modulo 360 (never exceeding a magnitude of the angle of $360^\circ$). So

$$-5\angle 36.86^\circ = 5\angle(36.86^\circ + 180^\circ) = 5\angle 216.86^\circ$$

or

$$-5\angle 36.86^\circ = 5\angle(36.86^\circ - 180^\circ) = 5\angle - 143.14^\circ$$

The same thing can be accomplished in rectangular form. Since

$$5\angle 36.86^\circ = 4 + j3$$

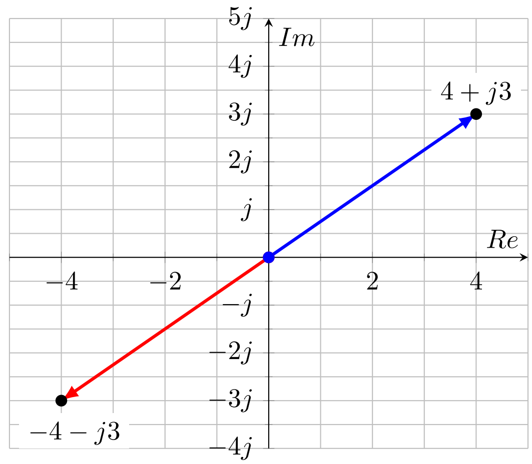
we can consider

$$-(4 + j3)$$

I included the parentheses deliberately to emphasis that the negative sign will distribute to both terms. Therefore,

$$-(4 + j3) = -4 - j3$$

Which we can see graphically as




We can also see graphically that the negation of the rectangular form matches the negation of the polar form. So

$$-4 - j3 = 5\angle 216.86°$$

*Subtraction* Subtraction is a simple combination of negation and addition. For instance to perform

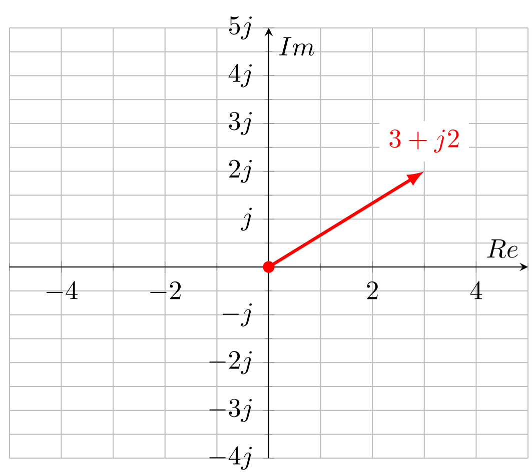
$$\begin{array}{rr} & 1+j4 \\ - & (3+j2) \\ \hline & -2+j2 \end{array}$$

Recall that the negative sign distributes to both terms before an addition is performed.

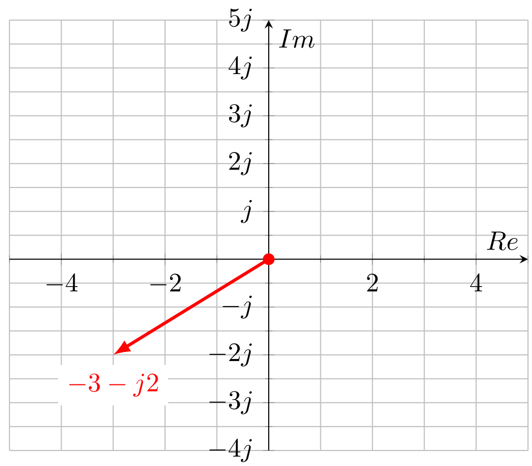
$$\begin{array}{rr} & 1+j4 \\ + & -3-j2 \\ \hline & -2+j2 \end{array}$$

Negate the second term, add it to the first. Let's consider this subtraction graphically.

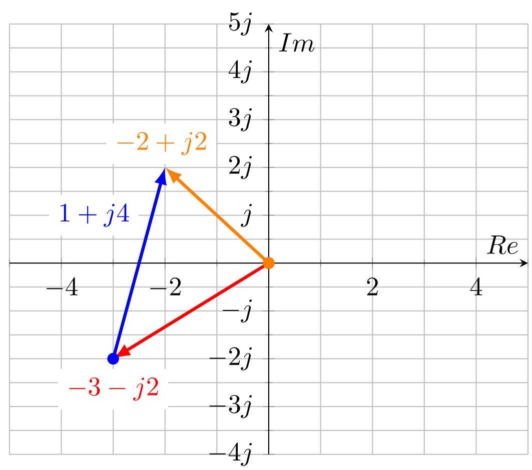
Start with the second term, $3 + j2$ in this case.



Next, negate it.




Finally, add the first operand, $1 + j4$ in this case, by placing the vectors tip-to-tail.



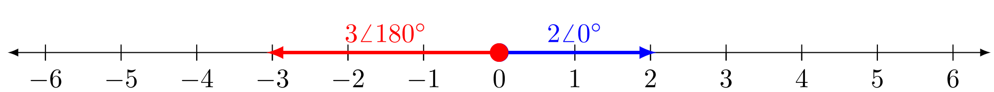

The result is shown in orange on the previous plot, $-2 + j2$.

*Multiplication* Multiplicaction stretches and rotates a value. This is true whether you are multiplying two scalar numbers or two complex numbers. When multiplying two numbers the result can be found using

$$(r_1\angle\theta_1)(r_2\angle\theta_2) = r_1 r_2\angle(\theta_1 + \theta_2)$$

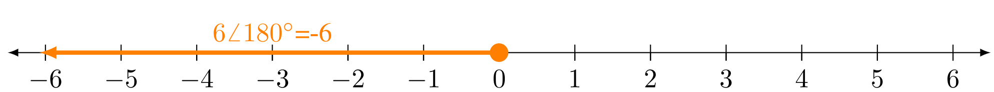
One magnitude is strecthed by the other. One angle is rotated by the other.

Lets's multiply two scalars. You were likely taught that a negative times a positive is a negative and a negative time a negative is a positive. This works fine for a single dimension but it is more general if restated as a strecth and a rotate. 2 times -3 can be shown on the number line though the vectors have been labeled in polar form.



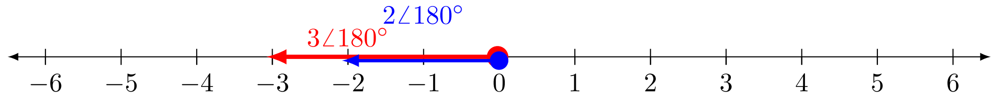

$2\angle0°$ is stretched to 3 times its length and rotated by $180°$. The result is $6\angle180°$ also know as -6.

$$(2 \times 3)\angle(0° + 180°) = 6\angle180° = -6$$



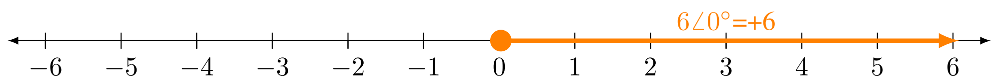

This result is consistent with the “negative times a positive is a negative” adage you were taught in grade school.

We can examine another case, -2 times -3, shown on the number line below with the value expressed as polar numbers.




$2\angle 180°$ is stretched to 3 times its length and rotated by $180°$. The result is $6\angle 360°$ which can be written as $6\angle 0°$ also know as +6.

$$(2 \times 3)\angle(180° + 180°) = 6\angle 360° = 6\angle 0° = +6$$




This result is consistent with the "negative times a negative is a positive" adage you were taught in grade school. So in polar form multiplication looks consistent with what you already know about multiplication of 1D values. The benefit of restating multiplication as stretching and rotating comes from extending it onto the 2D complex plane.

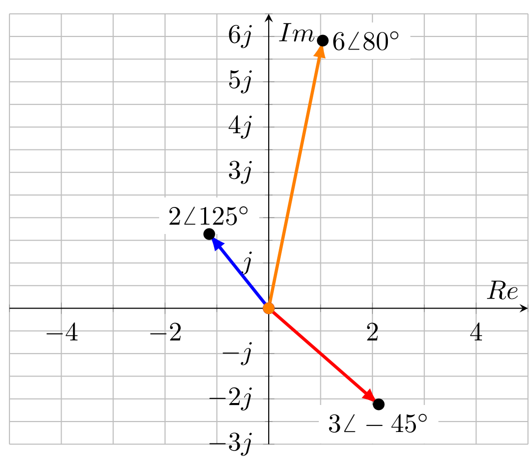
If we consider the example

$$(2\angle 125°)(3\angle -45°)$$

the result can be found by performing

$$(2 \times 3)\angle(125° + (-45°)) = 6\angle 80°$$

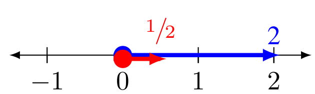
Graphically the two operands and the product are shown here



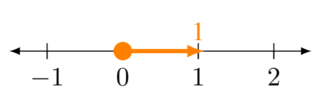

*Inverse* When a number is multiplied by its inverse the result is real valued 1 also known as $1\angle 0°$. With scalars, the inverse of 2 is $1/2$. Multiplying the two values results in

$$\frac{2}{1} \times \frac{1}{2} = \frac{2}{2} = 1 = 1\angle 0°$$

Graphically, the two vectors shown here




results in



when multiplied together.

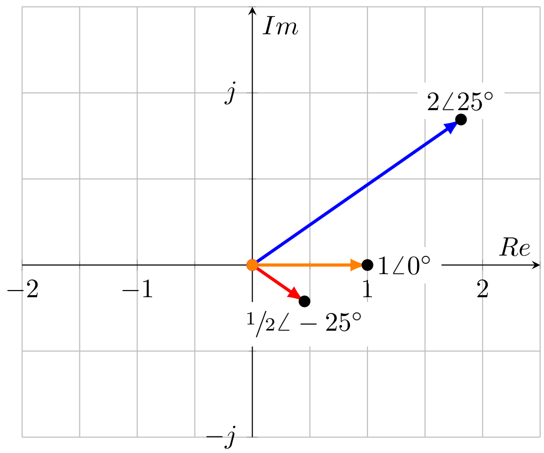
We can find the inverse of the polar number $r\angle\theta$ by asking 1) what magnitude will result in 1 when multiplied with $r$ and 2) what angle will result in $0°$ when added to $\theta$? The answer to both questions is

$$\frac{1}{r\angle\theta} = (r\angle\theta)^{-1} = \frac{1}{r}\angle{-\theta}$$

The multiplication of the value and its inverse can be expressed as

$$2\angle 25° \times (2\angle 25°)^{-1} = (2 \times \frac{1}{2})\angle(25° + (-25°)) = 1\angle 0° = 1$$

Graphically, $2\angle 25°$, its inverse, and the result of the multiplication are all shown here




Once again, the operation is consistent whether performed on a 1D number line or 2D complex plane.

*Division* Division is a simple combination of inversion and multiplication. For instance to perform

$$6 \div 2 = 3$$

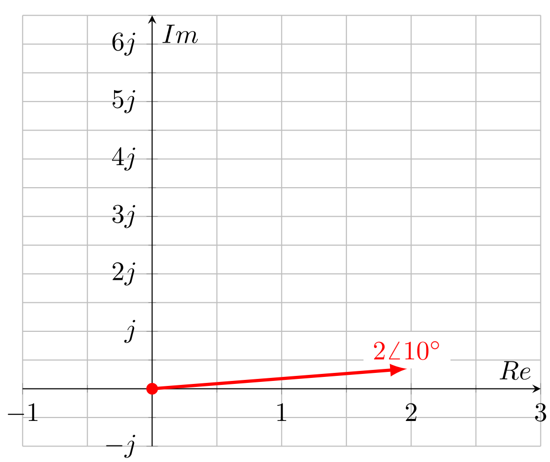
we invert the second operand and then perform a multiplication

$$6 \times \frac{1}{2} = 3$$

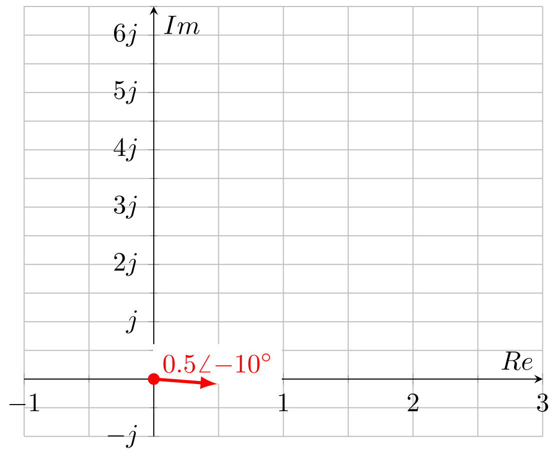
Let's perform this graphically on a complex example. First invert the second operand and then multiply it by the second. If the complex division is

$$(6\angle 75°) \div (2\angle 10°) = (3\angle 65°)$$

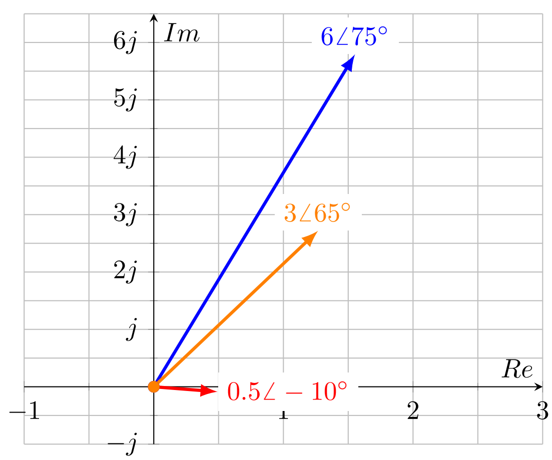
take the second operand



Invert it.




Finally, multiply it by the first operand



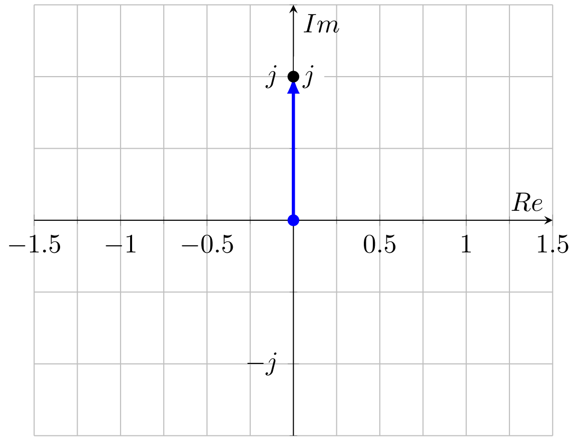

In general, complex division is performed as

$$(r_1\angle\theta_1) \div (r_2\angle\theta_2) = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

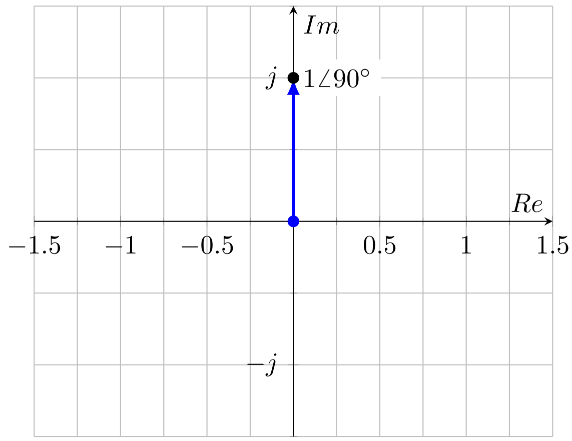
*So what's the deal with* $j = \sqrt{-1}$*?* You were undoubtedly taught the fact the the square root of a negative number does not exist and therefore it is labeled "imaginary". This is one of the greatest disservices that is done in math classrooms. Is it true the $\sqrt{-1} = j$? Yes. Is it the most important fact about imaginary (complex) numbers? No. It is a mere consequence of everything that we've discussed in this section. It is far more important that the operations performed on 1D scalar numbers are the same as the operations performed on 2D complex numbers. This is what should be taught. This should be the one thing most students remember about complex numbers. But it is not.

I said it was true so I will briefly demonstrate that using the operations and complex forms discussed in this section. But please remember that it is a consequence of the important concepts, not the important concept itself.

The complex value $j$ can be plotted as



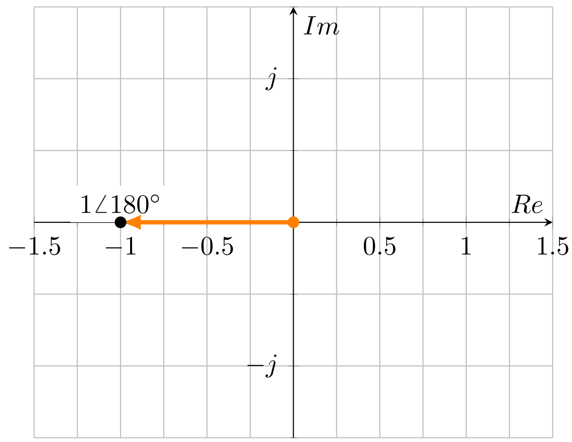

To locate that same point we can express it as $1\angle 90°$




Squaring that value we find

$$(1\angle 90°) \times (1\angle 90°) = (1 \times 1)\angle(90° + 90°) = 1\angle 180°$$

$1\angle 180°$ can also be expressed as -1




Reverting back to using $j$ for $1\angle 90°$ and -1 for $1\angle 180°$ we can state that

$$j \times j = j^2 = -1$$

and therefore

$$\sqrt{j^2} = j = \sqrt{-1}$$

So it's true but it is better thought of as a number, an actual number since it really exists just like any other point on the complex plane. It is a vector that when multiplying some other value it leaves that magnitude unchanged and rotates the angle by positive 90 degrees.

## 3.2 MATLAB and Complex Numbers

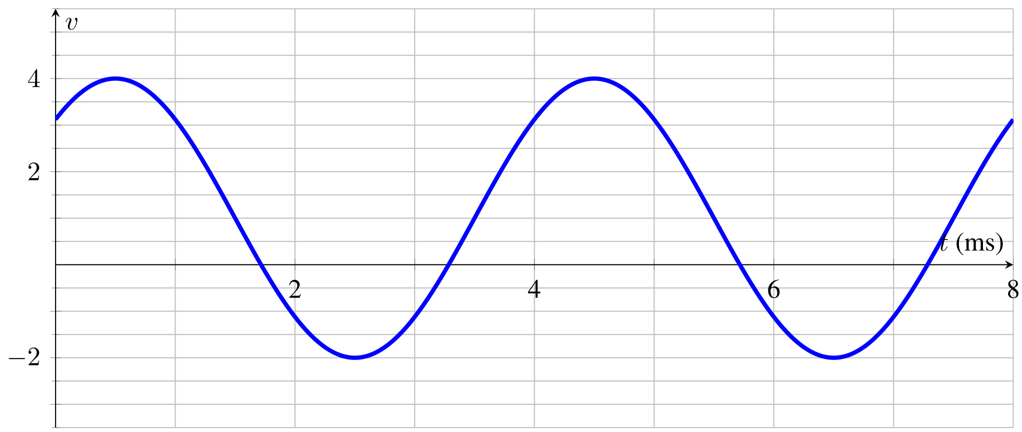
```
clear all %Clear all variables in the Workspace
close all %Close all open plots
clc % clear the Command Window
format short eng %Tell MATLAB to report values in fixed, Engineering notation

%% Assigning/displaying a rectangular value to/from a variable
%This line will display the result in the Command Window
a=(8-j*14)

%This line will not because it ends in a semicolon
b=(17+j*11);

%We can display the value using the variable name
b

%i and j are interchangable to indicate the imagniary number
c=4-i*18;

%% Assigning/displaying a polar value to/from a variable
%MATLAB stores complex values in rectangular form
%Anytime we use polar for we need to perform a conversion.

%6 at an angle of -5 degrees can be entered as
d=6*cos(-5*(pi/180))+j*6*sin(-5*(pi/180));
%this is roughly equal to 5.98-j0.523

%Notice the (pi/180) in the last line.  MATLAB deals in radians unless you
%tell it otherwise.  The previous line could be entered if you performed
%the assignment in radians explicitly
e=6*cos(-.08727)+j*6*sin(-.08727);
%Note that the two variables contain the same rectangular value.  Check
%this by typing the variable names in the command window or double clicking
%the variables in the Workspace window

%Alternatively you can use the exponential form
f=6*exp(j*-5*(pi/180));

%Displaying the values in rectangular form is as simple as typing the
%variable names in a script, the command window, or examining them in the
%Workspace window
a
b
c
d
e
f

%Displaying the values in polar form requires you to tell MATLAB that you'd
%like to see polar form.  You can request the magnitude of the complex
%value with
abs(a)
abs(b)
abs(c)
```

```
abs(d)
abs(e)
abs(f)

%And you can find the angle of a complex number using
angle(a)
angle(b)
angle(c)
angle(d)
angle(e)
angle(f)

%Note that each of the angles displayed above are in radians.  If you want
%them in degrees you need to perform the conversion by multiplying by
%(180/pi)
angle(a)*(180/pi)
angle(b)*(180/pi)
angle(c)*(180/pi)
angle(d)*(180/pi)
angle(e)*(180/pi)
angle(f)*(180/pi)

%So you can get the information out in polar form using abs() and angle()
%but it would be nice to put everything on the same line.  fprintf() allows
%you to do that
fprintf('%f at an angle of %f degrees\n',abs(f),angle(f)*(180/pi))
%I encourage you to type 'help fprintf' or 'doc fprintf' into the Command
%Window to learn more about output formatting

%% Performing operations on complex values
%Just perform the operation as you would expect.  The result will be
%stored, and displayed by default, in rectangular form regardless of how it
%was entered (rectangular, polar, or exponential)

%add
g=a+b
fprintf('%f at an angle of %f degrees\n',abs(g),angle(g)*(180/pi))

%subtract
h=a-b
fprintf('%f at an angle of %f degrees\n',abs(h),angle(h)*(180/pi))

%multiply
k=b*c
fprintf('%f at an angle of %f degrees\n',abs(k),angle(k)*(180/pi))

%divide
l=c/d
fprintf('%f at an angle of %f degrees\n',abs(l),angle(l)*(180/pi))

%exponent
m=a^e
fprintf('%f at an angle of %f degrees\n',abs(m),angle(m)*(180/pi))

%square root
n=sqrt(f)
```

```
fprintf('%f at an angle of %f degrees\n',abs(n),angle(n)*(180/pi))

%inversion
o=inv(g)
fprintf('%f at an angle of %f degrees\n',abs(o),angle(o)*(180/pi))
```

### 3.3 Relating the Time and Phasor Domains

It is often necessary to find a phasor representation of a signal based on a time-domain plot of the signal. This is typically done when using an oscilloscope to measure a sinusoidal signal. There are three piece of information to find with regards to a sinusoid: the amplitude, the frequency, and the phase shift. Let's consider the sinusoid shown below and consider how to find this information. I've included a forth value that can be found though it is not part of the sinusoid; the DC offset.




*Amplitude* The amplitude can be found by locating the maximum value, the minimum value, and subtracting them. This will give us the peak-to-peak amplitude. We can then convert this amplitude to any other form we wish. Examining the plot above we see that the maximum value is 4 V and the minimum value is -2 V. The peak-to-peak voltage is then found with

$$V_{pp} = 4 - (-2) = 6\ V_{pp}$$

Converting it to a peak voltage we find

$$V_p = \frac{V_{pp}}{2} = 3\ V_p$$

This is the value we write as the coefficient of the cosine when writing the time-domain function for this signal. It is therefore the magnitude of the polar phasor representation of the same signal.

*DC Offset* We can also find the DC offset with the maximum and minimum value of the sinusoid. Since the sinusoid is horizontally symmetrical we can simply average the two values to find the DC offset.

$$V_{off} = \frac{4 + (-2)}{2} = 1\ V$$

*Frequency/Period* In order to find the frequency of the sinusoid we first find the period. Find two points on the plot that are easy to identify on the time axis and are points where the signal begins to repeat. The maximum points are easy on our example above. The first peak is located at t=0.5 ms and the second is located at t=4.5 ms. The interval between these points define the time it takes for the phase to complete one cycle. We find this period with

$$T = 4.5 - 0.5 = 4\ ms$$

We find the frequency by inverting the period

$$f = \frac{1}{T} = \frac{1}{4\ ms} = 250\ Hz$$

From the frequency we can find the angular frequency

$$\omega = 2\pi f = 2\pi(250) = 1570.80\ {}^{rad}/_{s}$$

The angular frequency is the coefficient of $t$ when we write the time-domain sinusoid function.

*Phase Shift* The phase shift is usually measured from some reference along the time axis. With out any other reference we will use t=0 s. A cosine begins at its peak value when there is no phase shift, $\theta = 0°$. We've already located a peak value at 0.5 ms on the example above. Since this shift is to the right (in the positive direction) the shift has a negative sign, -0.5 ms. We already found that the signal has a periods of T=4 ms. Using this information we can find the phase shift with

$$\theta = \left(\frac{-0.5\ ms}{4\ ms}\right) 360° = -45°$$

*Putting it All Together* All of this information can be combined into two representations of the signal. The time-domain function can be written as

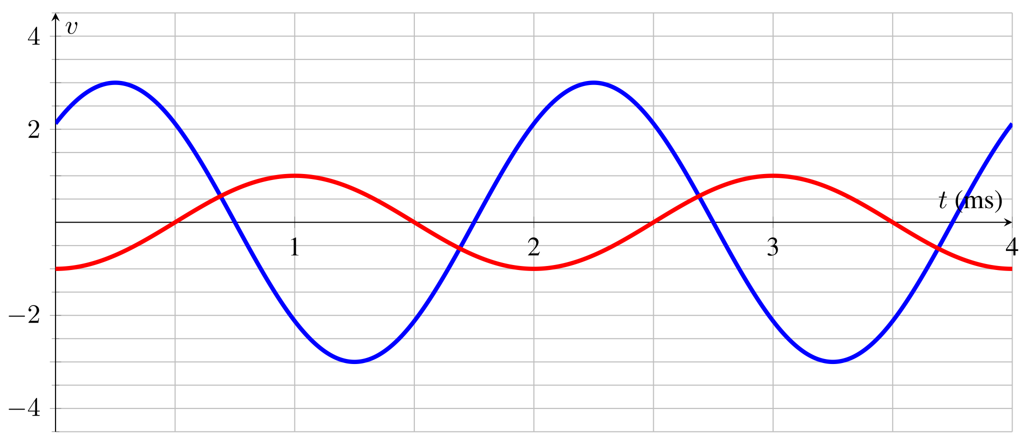
$$v(t) = 3\cos(1570.8t - 45°)\ V$$

The same signal can be represented by the phasor

$$V = 3\angle{-45°}\ V$$

### 3.3.1 Finding Phase Difference/Offset

We often need to find the difference in phase angle between two signals. When we measure values on an oscilloscope it is not clear what point to consider 0°. The source supplying power to a circuit is often taken as the reference and all phase angles are measured in reference to it. Let's try this on the signals shown here




First we find the period of the signals by finding two times where the signal begins to repeat. For instance, the red signal has positive-going zero-crossings at t=0.5 ms and t=2.5 ms. Therefore the period of that signal is

$$T = 2.5\ ms - 0.5\ ms = 2\ ms$$

We can confirm that blue signal has the same period by finding two points on that signal. They do not have to be the same positive-going zero-crossings. Here I can see that the blue signal has peak values at t=0.25 ms and t=2.25 ms leading to the period

$$T = 2.25\ ms - 0.25\ ms = 2\ ms$$

The period is the time that define 360 degrees of rotation for the phasors. We'll use this equivalence in a moment. Now we have to find two times that each signal is at the same point of rotation. I can look for negative-going zero-crossings, one of many possibilities. The blue signal crosses zero in the negative direction at t=0.75 ms and the red signal crosses zero in the negative direction at t=1.5 ms. Since the red signal crosses after we can say that is **lagging** the blue signal. The next thing we need to figure out is by how much does the red signal lag the blue signal. We find that with

$$\theta = \left(\frac{1.5 - 0.75}{2}\right) 360^\circ = 135^\circ$$

Therefore we can make two equivalent statements:

1. The red signal **lags** the blue signal by $135^\circ$
2. The blue signal **leads** the red signal by $135^\circ$

This information is important when considering some RF modulation schemes or the efficiency of power generation in an AC system.