Digital Logic Circuits 'Combinational Logic'

ELEC2200

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Boolean Algebra Examples

- $\overline{A+B\cdot C}$
- $=\overline{A}\cdot\overline{B\cdot C}$ (DeMorgan's)
- $=\overline{A}\cdot(\overline{B}+\overline{C})$ (DeMorgan's)
- $=\overline{A}\cdot\overline{B}+\overline{A}\cdot\overline{C}$ (Distributivity)

- Both expressions are implementations of the same switching function

Boolean Algebra Examples

- $\overline{A \cdot (B+C) + A' \cdot B}$
- $\overline{A \cdot B + A \cdot C + A' \cdot B}$ (Distributivity)
- $\overline{B + A \cdot C}$ (Absorption)
- $\overline{B} \cdot \overline{A \cdot C}$ (DeMorgan's)
- $\overline{B} \cdot (\overline{A} + \overline{C})$ (DeMorgan's)
- $\overline{B} \cdot \overline{A} + \overline{B} \cdot \overline{C}$ (Distributivity)

Boolean Algebra Examples

- $A \cdot B \cdot C + \overline{A} \cdot D + \overline{B} \cdot D + C \cdot D$
- $A \cdot B \cdot C + (\overline{A} + \overline{B}) \cdot D + C \cdot D$ (Absorption)
- $A \cdot B \cdot C + \overline{A \cdot B} \cdot D + C \cdot D$ (DeMorgan's)
- $A \cdot B \cdot C + \overline{A \cdot B} \cdot D$ (Consensus)
- $A \cdot B \cdot C + (\overline{A} + \overline{B}) \cdot D$ (DeMorgan's)
- $A \cdot B \cdot C + \overline{A} \cdot \overline{D} + \overline{B} \cdot \overline{D}$ (Absorption)

Boolean Algebra

- Application of Boolean algebra does simplify expressions
- Requires some creativity, some vigilance, and a little luck
- Can a better method be developed?

Algebraic Forms

- Literal: A variable or its inverse

$$A, B, C, \overline{A}, B', \overline{C}$$

- Product term: Two literals joined by an AND

$$A \cdot B, \overline{B} \cdot C, B \cdot \overline{C}, \overline{A} \cdot \overline{C}$$

- Sum term: Two literals joined by an OR

$$A+B, \overline{B}+C, B+\overline{C}, \overline{A}+\overline{C}$$

- Sum of Products(SOP): Multiple product terms joined by ORs

$$A \cdot B+\overline{B} \cdot C+B \cdot \overline{C}+\overline{A} \cdot \overline{C}$$

- Product of Sums(POS): Multiple sum terms joined by ANDs

$$(A+B) \cdot (\overline{B}+C) \cdot (B+\overline{C}) \cdot (\overline{A}+\overline{C})$$

Canonical Algebraic Forms

- Minterm: A product term in which each variable, or its complement, appears exactly once
- Maxterm: A sum term in which each variable, or its complement, appears exactly once
- Canonical SOP: A SOP expressed using only minterms
- Canonical POS: A POS expressed using only maxterms

Canonical Algebraic Forms

- Canonical SOP Example:

$$f(A,B,C)=A\cdot B\cdot C+A\cdot \overline{B}\cdot C+A\cdot B\cdot \overline{C}+\overline{A}\cdot B\cdot \overline{C}$$

- Canonical POS Example:

$$f(A,B,C)=(A+B+C)\cdot(A+\overline{B}+C)\cdot(A+B+\overline{C})\cdot(\overline{A}+B+\overline{C})$$

- Canonical expressions remove ambiguity in representing a particular switching function
    - There is only one canonical expression of a switching function
- Canonical expressions allow for simplified expression with a familiar form

Canonical Algebraic Forms

- Examining our switching functions (n=2) and selecting $f_B(A,B)$ for an example

A	B	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_A	f_B	f_C	f_D	f_E	f_F
0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

- Each input case that produces an output of 1 can be expressed as an individual minterm
- $f_B(A,B)=\overline{A}\cdot\overline{B}+\overline{A}\cdot B+A\cdot B$
- $f_B(A,B)=\sum m(0,1,3)$

Canonical Algebraic Forms

- $f_B(A,B)=\overline{A}\cdot\overline{B}+\overline{A}\cdot B+A\cdot B$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A}\cdot\overline{B}$	$\overline{A}\cdot B$	$A\cdot B$	$\overline{A}\cdot\overline{B}+\overline{A}\cdot B+A\cdot B$
0	0	1	1	1	0	0	1
0	1	1	0	0	1	0	1
1	0	0	1	0	0	0	0
1	1	0	0	0	0	1	1

Canonical Algebraic Forms

- Minterms for f(A,B,C)

Row #	Inputs	Minterm #	Minterm
0	000	m(0)	$\overline{A}\cdot\overline{B}\cdot\overline{C}$
1	001	m(1)	$\overline{A}\cdot\overline{B}\cdot C$
2	010	m(2)	$\overline{A}\cdot B\cdot\overline{C}$
3	011	m(3)	$\overline{A}\cdot B\cdot C$
4	100	m(4)	$A\cdot\overline{B}\cdot\overline{C}$
5	101	m(5)	$A\cdot\overline{B}\cdot C$
6	110	m(6)	$A\cdot B\cdot\overline{C}$
7	111	m(7)	$A\cdot B\cdot C$

Canonical Algebraic Forms

- Examining our switching functions (n=2) and selecting $f_2(A,B)$ for an example

A	B	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_A	f_B	f_C	f_D	f_E	f_F
0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

- Each input case that produces an output of 0 can be expressed as an individual maxterm
- $f_2(A,B)=(A+B)\cdot(\overline{A}+B)\cdot(\overline{A}+\overline{B})$
- $f_2(A,B)=\Pi M(0,2,3)$

Canonical Algebraic Forms

- $f_2(A,B)=(A+B)\cdot(\overline{A}+B)\cdot(\overline{A}+\overline{B})$

A	B	$\overline{A}$	$\overline{B}$	$A+B$	$\overline{A}+B$	$\overline{A}+\overline{B}$	$(A+B)\cdot(\overline{A}+B)\cdot(\overline{A}+\overline{B})$
0	0	1	1	0	1	1	0
0	1	1	0	1	1	1	1
1	0	0	1	1	0	1	0
1	1	0	0	1	1	0	0

Canonical Algebraic Forms

- Maxterms for f(A,B,C)

Row #	Inputs	Maxterm #	Maxterm
0	000	M(0)	$A+B+C$
1	001	M(1)	$A+B+\overline{C}$
2	010	M(2)	$A+\overline{B}+C$
3	011	M(3)	$A+\overline{B}+\overline{C}$
4	100	M(4)	$\overline{A}+B+C$
5	101	M(5)	$\overline{A}+B+\overline{C}$
6	110	M(6)	$\overline{A}+\overline{B}+C$
7	111	M(7)	$\overline{A}+\overline{B}+\overline{C}$

Canonical Forms

- What is the canonical SOP expression for $f(A,B,C)=\sum m(1,2,5,7)$?

- What is the canonical POS expression for $f(A,B,C)=\prod M(0,3,4,6)$?

Canonical Forms

- What is the canonical SOP expression for $f(A,B,C)=\sum m(1,2,5,7)$?
  - $f(A,B,C)=\overline{A}\cdot\overline{B}\cdot C+\overline{A}\cdot B\cdot\overline{C}+A\cdot\overline{B}\cdot C+A\cdot B\cdot C$

- What is the canonical POS expression for $f(A,B,C)=\prod M(0,3,4,6)$?
  - $f(A,B,C)=(A+B+C)\cdot(A+\overline{B}+\overline{C})\cdot(\overline{A}+B+C)\cdot(\overline{A}+\overline{B}+C)$

Minimization

- We have looked at heuristic methods
  - Boolean Algebra
  - May result in suboptimal expression
- Now a look at methodical methods
  - **Karnaugh maps**
  - Quine-McCluskey method
  - Petrick's method
  - Generalized Consensus
  - More consistent, optimal results

Minimization

- Our goal: find an SOP/POS expression that contains the fewest terms and literals for a particular switching function
- Minimum SOP(MSOP): fewest product terms
- Minimum POS(MPOS): fewest sum terms

Karnaugh Maps

- Method of minimization
- Useful for up to 6 variables
- An n-variable K-map has 2^n cells that correspond to the 2^n rows of a truth table
- Arrangement of cells is important
- Output of switching function is then used to fill in cells

Karnaugh Maps

- The 2 variable K-map

A\B	0	1
0	Z_0	Z_1
1	Z_2	Z_3

Row	A	B	Z
0	0	0	Z_0
1	0	1	Z_1
2	1	0	Z_2
3	1	1	Z_3

- Cell are positioned so that adjacent cells correspond to a one-bit change in input

Karnaugh Maps

- 3 variable K-map

AB\C	0	1
00	Z_0	Z_1
01	Z_2	Z_3
11	Z_6	Z_7
10	Z_4	Z_5

Row	A	B	C	Z
0	0	0	0	Z_0
1	0	0	1	Z_1
2	0	1	0	Z_2
3	0	1	1	Z_3
4	1	0	0	Z_4
5	1	0	1	Z_5
6	1	1	0	Z_6
7	1	1	1	Z_7

- Note reordering in K-Map
- Note wrapping of 1-bit change

Karnaugh Maps

- 4 variable K-map

AB\CD	00	01	11	10
00	Z_0	Z_1	Z_3	Z_2
01	Z_4	Z_5	Z_7	Z_6
11	Z_{12}	Z_{13}	Z_{15}	Z_{14}
10	Z_8	Z_9	Z_{11}	Z_{10}

5 variable K-map

A=0

BC\DE	00	01	11	10
00	Z_0	Z_1	Z_3	Z_2
01	Z_4	Z_5	Z_7	Z_6
11	Z_{12}	Z_{13}	Z_{15}	Z_{14}
10	Z_8	Z_9	Z_{11}	Z_{10}

A=1

BC\DE	00	01	11	10
00	Z_{16}	Z_{17}	Z_{19}	Z_{18}
01	Z_{20}	Z_{21}	Z_{23}	Z_{22}
11	Z_{28}	Z_{29}	Z_{31}	Z_{30}
10	Z_8	Z_9	Z_{11}	Z_{10}

Karnaugh Maps

- 6 variable K-map

A=0 B=0

CD\EF	00	01	11	10
00	Z_0	Z_1	Z_3	Z_2
01	Z_4	Z_5	Z_7	Z_6
11	Z_{12}	Z_{13}	Z_{15}	Z_{14}
10	Z_8	Z_9	Z_{11}	Z_{10}

A=0 B=1

CD\EF	00	01	11	10
00	Z_{16}	Z_{17}	Z_{19}	Z_{18}
01	Z_{20}	Z_{21}	Z_{23}	Z_{22}
11	Z_{28}	Z_{29}	Z_{31}	Z_{30}
10	Z_{24}	Z_{25}	Z_{27}	Z_{26}

A=1 B=1

CD\EF	00	01	11	10
00	Z_{48}	Z_{49}	Z_{51}	Z_{50}
01	Z_{52}	Z_{53}	Z_{55}	Z_{54}
11	Z_{60}	Z_{61}	Z_{63}	Z_{62}
10	Z_{56}	Z_{57}	Z_{59}	Z_{58}

A=1 B=0

CD\EF	00	01	11	10
00	Z_{32}	Z_{33}	Z_{35}	Z_{34}
01	Z_{36}	Z_{37}	Z_{39}	Z_{38}
11	Z_{44}	Z_{45}	Z_{47}	Z_{46}
10	Z_{40}	Z_{41}	Z_{43}	Z_{42}

Karnaugh Maps

- Let's populate a K-map

  f(A,B,C,D)=

  ∑m(1,2,3,6,8,9,10,12,13,14)

AB\CD	00	01	11	10
00				
01				
11				
10				

Row	A	B	C	D	Z
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	1
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	0
8	1	0	0	0	1
9	1	0	0	1	1
10	1	0	1	0	1
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	1
14	1	1	1	0	1
15	1	1	1	1	0

Karnaugh Maps

- Let's populate a K-map

f(A,B,C,D)=

$\sum m(1,2,3,6,8,9,10,12,13,14)$

AB\CD	00	01	11	10
00		1	1	1
01				1
11	1	1		1
10	1	1		1

Row	A	B	C	D	Z
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	1
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	0
8	1	0	0	0	1
9	1	0	0	1	1
10	1	0	1	0	1
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	1
14	1	1	1	0	1
15	1	1	1	1	0

Karnaugh Maps

- We now group the cells with 1's together
- Taking a simple group:
- The two minterms associated with this group are:

AB\CD	00	01	11	10
00		1	1	1
01				1
11	1	1		1
10	1	1		1

- $\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot D+\overline{A}\cdot\overline{B}\cdot C\cdot D$

Karnaugh Maps

- Recall Shannon's Expansion Theorem
- $f(A,B,C,D)=C\cdot f(A,B,1,D)+\overline{C}\cdot f(A,B,0,D)$
- $\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot D+\overline{A}\cdot\overline{B}\cdot C\cdot D$
- $\overline{C}\cdot\overline{A}\cdot\overline{B}\cdot D+C\cdot\overline{A}\cdot\overline{B}\cdot D$
- $(\overline{C}+C)\cdot\overline{A}\cdot\overline{B}\cdot D$
- $\overline{A}\cdot\overline{B}\cdot D$
- This is minimal product term for this group

AB\CD	00	01	11	10
00		1	1	1
01				1
11	1	1		1
10	1	1		1

Karnaugh Maps

- Applying this to other groups
- $\overline{A}\cdot\overline{B}\cdot C\cdot\overline{D}+\overline{A}\cdot B\cdot C\cdot\overline{D}$
- $\overline{A}\cdot C\cdot\overline{D}$
- $A\cdot B\cdot C\cdot\overline{D}+A\cdot\overline{B}\cdot C\cdot\overline{D}$
- $A\cdot C\cdot\overline{D}$
- $\overline{A}\cdot C\cdot\overline{D}+A\cdot C\cdot\overline{D}$
- $(\overline{A}+A)\cdot C\cdot\overline{D}$
- $C\cdot\overline{D}$ <-Minimal product term for two groups together

AB\CD	00	01	11	10
00		1	1	1
01				1
11	1	1		1
10	1	1		1

Karnaugh Maps

- Rules for grouping K-map cells:
  - Groups must contain 2^n cells
  - Groups must be rectangular
  - All cells must contain equivalent values (1,0,d)
  - A group must cover a cell that is not covered by any other group
  - The larger the groups, the simpler the expression
  - Fewer groups results in fewer terms

Karnaugh Maps

- Another look at our example
- 3 groups=3 terms
- $\overline{A}\cdot\overline{B}\cdot D$
- $C\cdot\overline{D}$
- $A\cdot\overline{C}$
- $f(A,B,C,D)=\sum m(1,2,3,6,8,9,10,12,13,14)$
- $f(A,B,C,D)=\overline{A}\cdot\overline{B}\cdot\overline{C}\cdot D+\overline{A}\cdot\overline{B}\cdot C\cdot D+\overline{A}\cdot\overline{B}\cdot C\cdot\overline{D}+\overline{A}\cdot B\cdot C\cdot\overline{D}+A\cdot B\cdot C\cdot\overline{D}+A\cdot\overline{B}\cdot C\cdot\overline{D}+A\cdot B\cdot\overline{C}\cdot\overline{D}+A\cdot B\cdot\overline{C}\cdot D+A\cdot\overline{B}\cdot\overline{C}\cdot\overline{D}+A\cdot\overline{B}\cdot\overline{C}\cdot D$
- $f(A,B,C,D)=\overline{A}\cdot\overline{B}\cdot D+C\cdot\overline{D}+A\cdot\overline{C}$

AB\CD	00	01	11	10
00		1	1	1
01				1
11	1	1		1
10	1	1		1

Karnaugh Maps

- If you can't see the expression corresponding to a group:
- Write all input combinations, order doesn't matter
- If column contains multiple values, cross it out
- Use what's left to form term

AB\CD	00	01	11	10
00		1	1	1
01				1
11	1	1		1
10	1	1		1

A	B	C	D
0	0	0	1
0	0	1	1

$\rightarrow \overline{A} \cdot \overline{B} \cdot D$

A	B	C	D
0	0	1	0
0	1	1	0
1	1	1	0
1	0	1	0

$\rightarrow C \cdot \overline{D}$

A	B	C	D
1	1	0	0
1	1	0	1
1	0	0	0
1	0	0	1

$\rightarrow A \cdot \overline{C}$

Karnaugh Maps

- We can minimize POS expressions too
- f(A,B,C,D)=ΠM(0,2,8,10,12,14)
- Maxterm, fill in zeros

AB\CD	00	01	11	10
00	0			0
01				
11	0			0
10	0			0

Karnaugh Maps

- We can minimize POS expressions too
- f(A,B,C,D)=ΠM(0,2,8,10,12,14)
- Worst case, 6 groups

AB\CD	00	01	11	10
00	0			0
01				
11	0			0
10	0			0

- $f(A,B,C,D)=(A+B+C+D)\cdot(A+B+\overline{C}+D)\cdot(\overline{A}+\overline{B}+C+D)\cdot(\overline{A}+B+C+D)\cdot(\overline{A}+\overline{B}+\overline{C}+D)\cdot(\overline{A}+B+\overline{C}+D)$
- This is the canonical POS expression

Karnaugh Maps

- We can minimize POS expressions too
- $f(A,B,C,D)=\Pi M(0,2,8,10,12,14)$
- A reasonable guess, 4 groups

AB\CD	00	01	11	10
00	0			0
01				
11	0			0
10	0			0

- $f(A,B,C,D)=(A+B+C+D)\cdot(A+B+\overline{C}+D)\cdot(\overline{A}+C+D)\cdot(\overline{A}+\overline{C}+D)$
- An improvement, but we can do better

Karnaugh Maps

- We can minimize POS expressions too
- $f(A,B,C,D)=\Pi M(0,2,8,10,12,14)$
- Remember wrapping, 2 groups!

AB\CD	00	01	11	10
00	0			0
01				
11	0			0
10	0			0

- $f(A,B,C,D)=(B+D)\cdot(\overline{A}+D)$
- This is the MPOS expression

Karnaugh Map

- All expressions implement the same switching function
- $f(A,B,C,D)=(A+B+C+D)\cdot(A+B+\overline{C}+D)\cdot(\overline{A}+\overline{B}+C+D)\cdot(\overline{A}+B+C+D)\cdot(\overline{A}+\overline{B}+\overline{C}+D)\cdot(\overline{A}+B+\overline{C}+D)$
- $f(A,B,C,D)=(A+B+C+D)\cdot(A+B+\overline{C}+D)\cdot(\overline{A}+C+D)\cdot(\overline{A}+\overline{C}+D)$
- $f(A,B,C,D)=(B+D)\cdot(\overline{A}+D)$
- Simplicity=Lower cost, faster response, smaller circuit, etc...
- We'll define these metrics more formally shortly

It's Okay to Not Care

- Sometimes input combinations are of no concern
  - They may not exist.
  - BCD uses only 10 of 16 possible inputs
- If we don't care, output can be either a 1 or a 0
- We can choose whichever one results in the simplest expression

'Don't Cares'

- How do we denote a don't care condition
  - Minterm: $f(A,B,C)=\sum m(1,3,6,7)+d(2)$
  - Maxterm: $f(A,B,C)=\prod M(0,4,5)+d(2)$
- In truth tables and K-maps
  - x,d,-,2 represent a don't care condition
  - x is probably most common
- Let's look at the minterm expresison

'Don't Care'

- $Z=f(A,B,C)=\sum m(1,3,6,7)+d(2)$

A	B	C	Z
0	0	0	
0	0	1	1
0	1	0	x
0	1	1	1
1	0	0	
1	0	1	
1	1	0	1
1	1	1	1

A\BC	00	01	11	10
0		1	1	x
1			1	1

'Don't Care'

- If we don't use don't care condition
  - $\overline{A} \cdot C + A \cdot B$
  - Two terms
  - 4 literals

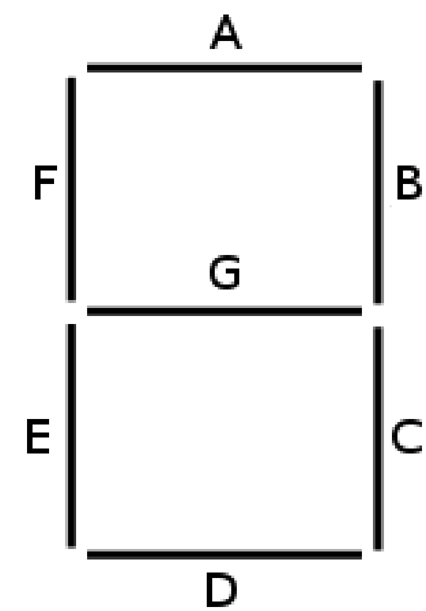
A\BC	00	01	11	10
0		1	1	x
1			1	1

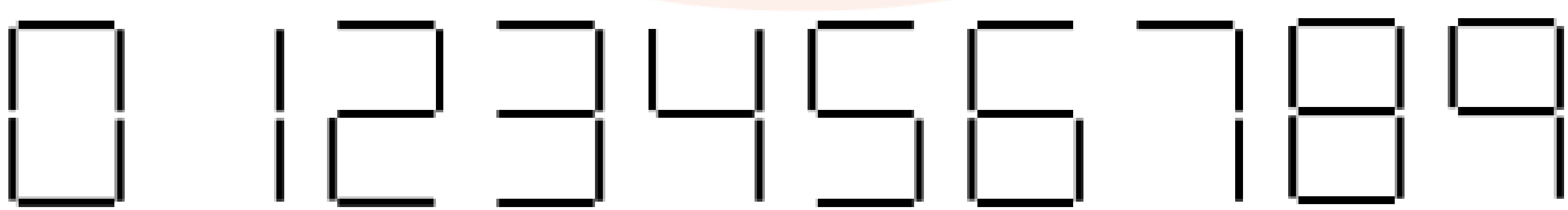
- If we include the don't care cell
  - $\overline{A} \cdot C + B$
  - Still two terms
  - 3 literals

A\BC	00	01	11	10
0		1	1	x
1			1	1

BCD → 7 Segment

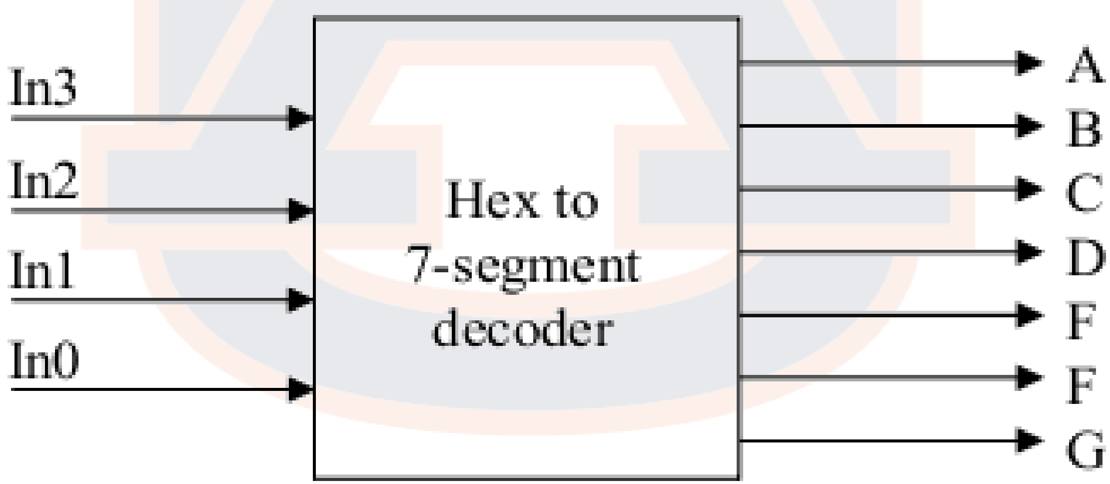
- Common Display Method
- A segment is 'on' if a '1' is present
- 4 inputs (BCD value)
- 7 outputs (Segment value)
- 7 individual expressions





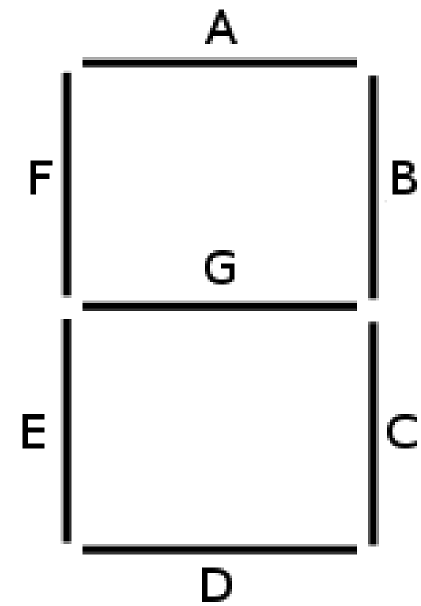
BCD → 7 Segment

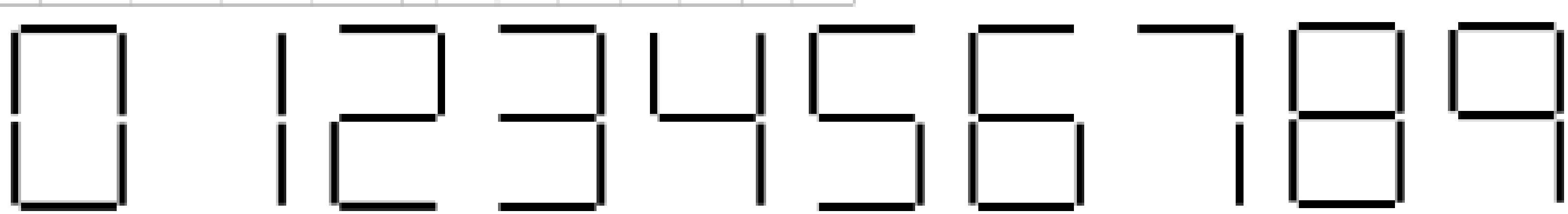
- All inputs shared by all 7 circuits
- Each circuit → One truth table column
- Each circuit → One K-map



BCD → 7 Segment

Digit	In3	In2	In1	In0	a	b	c	d	e	f	g
0	0	0	0	0	1	1	1	1	1	1	0
1	0	0	0	1	0	1	1	0	0	0	0
2	0	0	1	0	1	1	0	1	1	0	1
3	0	0	1	1	1	1	1	1	0	0	1
4	0	1	0	0	0	1	1	0	0	1	1
5	0	1	0	1	1	0	1	1	0	1	1
6	0	1	1	0	1	0	1	1	1	1	1
7	0	1	1	1	1	1	1	0	0	0	0
8	1	0	0	0	1	1	1	1	1	1	1
9	1	0	0	1	1	1	1	0	0	1	1
A	1	0	1	0	x	x	x	x	x	x	x
B	1	0	1	1	x	x	x	x	x	x	x
C	1	1	0	0	x	x	x	x	x	x	x
D	1	1	0	1	x	x	x	x	x	x	x
E	1	1	1	0	x	x	x	x	x	x	x
F	1	1	1	1	x	x	x	x	x	x	x





BCD → 7 Segment

- After we have expressions we must
  - Draw logic diagram
  - Analyze size and performance
  - Simulate for design validation
  - Optimize
  - Re-simulate optimized desing